SIMPLE LIE COLOR ALGEBRAS OF WITT-TYPE

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Abstract. Let K be a field and let $\varepsilon\colon \Gamma \times \Gamma \to K^\bullet$ be a bicharacter defined on the multiplicative group Γ. We suppose that A is a Γ-graded, associative K-algebra which is color commutative with respect to ε. Furthermore, let Δ be a nonzero Γ-graded, K-vector space of color derivations of A and suppose that Δ is also color commutative with respect to the bicharacter ε. Then, with a rather natural definition, $A \otimes_K \Delta = A\Delta$ becomes a Lie color algebra, and we obtain necessary and sufficient conditions here for this Lie color algebra to be simple. With two minor exceptions when $\dim_K \Delta = 1$, simplicity occurs if and only if A is graded Δ-simple and $A^\Delta \otimes \Delta = A^\Delta \Delta$ acts faithfully as color derivations on A.

§1. Construction of the Lie Algebra

In the recent paper [Pa], we took a ring theoretic approach to the construction of simple Lie algebras of Witt-type. Here, we use similar methods to show that analogous results hold in the context of Lie color algebras. Since the color aspects of this construction may be new, we will include all definitions and carefully verify all required identities.

To start with, let K be a field and let Γ be a multiplicative abelian group. Recall that a bicharacter $\varepsilon\colon \Gamma \times \Gamma \to K^\bullet$ is a map which satisfies

E1. $\varepsilon(x, y)\varepsilon(y, x) = 1$ for all $x, y \in \Gamma$
E2. $\varepsilon(x, yz) = \varepsilon(x, y)\varepsilon(x, z)$ for all $x, y, z \in \Gamma$.

Furthermore, it is clear that these imply

E3. $\varepsilon(xy, z) = \varepsilon(x, z)\varepsilon(y, z)$ for all $x, y, z \in \Gamma$.

We will fix K, Γ and ε throughout this work.

Now let A be a Γ-graded associative K-algebra. Thus $A = \oplus \sum_{x\in\Gamma} A_x$ is a direct sum of the K-subspaces A_x, indexed by the elements $x \in \Gamma$, and these summands

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satisfy $A_xA_y \subseteq A_{xy}$ for all $x, y \in \Gamma$. Of course, $1 \in A_1$. As usual, the homogeneous elements $\mathcal{H}(A)$ of A are precisely the *nonzero* elements in the various components A_x. Thus each homogeneous element $a \in \mathcal{H}(A)$ determines a unique group element $a^\# \in \Gamma$ by $a \in A_{a^\#}$. Fortunately, we can almost always drop the symbol $^\#$ here, since confusion rarely occurs. In particular, if a is homogeneous, then we have $a \in A_a$ and also $aA_y \subseteq A_{ay}$ for all $y \in \Gamma$.

As is well known (see [B, page 15]), the K-linear map $[\ ,\ ]\colon A \times A \to A$ defined by $[a, b] = ab - \varepsilon(a, b)ba$ for all $a, b \in \mathcal{H}(A)$, gives rise to a Lie color algebra structure on A. Here, of course, we have abbreviated $\varepsilon(a^\#, b^\#)$ by $\varepsilon(a, b)$. We will actually require very little about this color algebra, since the appropriate graded analog of the commutativity of A in [Pa] is that A be color commutative. By this we mean that $[A, A] = 0$ or equivalently that

C1. $ab = \varepsilon(a, b)ba$ for all $a, b \in \mathcal{H}(A)$.

Lemma 1.1. *If A is a color commutative, Γ-graded algebra, and if V is a graded subset of A, then $AV = VA$. In particular,*

i. *If V is a right or left ideal of A, then V is a two-sided ideal of A.*
ii. *If V is nilpotent, then VA is a nilpotent ideal of A.*
iii. *If $\mathcal{H}(V)$ consists of nilpotent elements, then VA is a nil ideal of A.*

Proof. If $x, y \in \Gamma$, then $V_xA_y = \varepsilon(x, y)A_yV_x = A_yV_x$. Thus, since $V = \sum_{x\in\Gamma} V_x$ and $A = \sum_{y\in\Gamma} A_y$, we conclude that $VA = AV$. Parts (i) and (ii) are now obvious, and (iii) follows since $VA = \sum_{v\in\mathcal{H}(V)} vA$ and each vA is a nilpotent ideal of A. □

Next, a nonzero K-linear map $\partial\colon A \to A$ is said to be a homogeneous color derivation of grade $\partial^\# \in \Gamma$ if

D1. $\partial(A_x) \subseteq A_{\partial x}$ for all $x \in \Gamma$
D2. $\partial(ab) = \partial(a)b + \varepsilon(\partial, a)a\partial(b)$ for all $a, b \in \mathcal{H}(A)$.

As usual, we dropped the $^\#$ symbol in the above formulas, so that $A_{\partial x}$ is really $A_{\partial^\# x}$ and similarly $\varepsilon(\partial, a) = \varepsilon(\partial^\#, a^\#)$. Of course, when $a = b = 1$, (D2) implies that $\partial(1) = 0$, and hence

D3. $\partial(K) = 0$.

For convenience, we denote the K-linear span of all such homogeneous derivations by $\mathrm{Der}^\varepsilon(A)$, so that $\mathrm{Der}^\varepsilon(A) = \oplus\sum_{x\in\Gamma} \mathrm{Der}^\varepsilon_x(A)$, where $\mathrm{Der}^\varepsilon_x(A)$ is the K-subspace consisting of all such derivations of grade $x \in \Gamma$ along with the derivation 0. Note that the directness of the sum follows immediately from (D1). Now, if $\alpha, \beta \in \mathcal{H}(\mathrm{Der}^\varepsilon(A))$, then $[\alpha, \beta] = \alpha\beta - \varepsilon(\alpha, \beta)\beta\alpha$ is easily seen to be either 0 or a homogeneous color derivation of grade $\alpha^\#\beta^\#$. In particular, the K-linear extension of this bracket yields a map $[\ ,\ ]\colon \mathrm{Der}^\varepsilon(A) \times \mathrm{Der}^\varepsilon(A) \to \mathrm{Der}^\varepsilon(A)$, and in this way, $\mathrm{Der}^\varepsilon(A)$ becomes a Lie color algebra.

Lemma 1.2. *Assume that A is a color commutative, Γ-graded algebra. Let $a, b \in \mathcal{H}(A)$ and let $\alpha, \beta \in \mathcal{H}(\mathrm{Der}^{\varepsilon}(A))$.*

i. *The map $a\alpha\colon A \to A$ given by $a\alpha\colon r \mapsto a\alpha(r)$ for all $r \in A$ is either 0 or a homogeneous color derivation of A of grade $a^{\#}\alpha^{\#}$.*
ii. *If $[\alpha, \beta] = 0$, then $[a\alpha, b\beta] = a\alpha(b)\beta - \varepsilon(a\alpha, b\beta)b\beta(a)\alpha$ as operators on A and hence as elements of $\mathrm{Der}^{\varepsilon}(A)$.*

Proof. (i) We can suppose that $a\alpha \neq 0$. If $r, s \in \mathcal{H}(A)$, then the color commutativity of A yields $ar = \varepsilon(a, r)ra$ and hence

$$\begin{aligned}(a\alpha)(rs) &= a\alpha(r)s + a\varepsilon(\alpha, r)r\alpha(s)\\ &= a\alpha(r)s + \varepsilon(a, r)\varepsilon(\alpha, r)ra\alpha(s)\\ &= (a\alpha)(r)s + \varepsilon(a\alpha, r)r(a\alpha)(s)\end{aligned}$$

for all $r, s \in \mathcal{H}(A)$. Thus, since $(a\alpha)A_x \subseteq aA_{\alpha x} \subseteq A_{a\alpha x}$ for all $x \in \Gamma$, $a\alpha$ is indeed a homogeneous color derivation of A of grade $a^{\#}\alpha^{\#}$.

(ii) Again, we can assume that $a\alpha$ and $b\beta$ are nonzero. If $r \in \mathcal{H}(A)$, then

$$\begin{aligned}(a\alpha)(b\beta)(r) &= a\alpha(b)\beta(r) + ab\varepsilon(\alpha, b)(\alpha\beta)(r)\\ (b\beta)(a\alpha)(r) &= b\beta(a)\alpha(r) + ba\varepsilon(\beta, a)(\beta\alpha)(r).\end{aligned}$$

Now A is color commutative, so $ab = \varepsilon(a, b)ba$ and $[\alpha, \beta] = 0$ implies that $\alpha\beta = \varepsilon(\alpha, \beta)\beta\alpha$. Thus $ab\varepsilon(\alpha, b)(\alpha\beta)(r) = ba\varepsilon(\beta, a)(\beta\alpha)(r)\cdot k$ where

$$k = \varepsilon(a, b)\varepsilon(\alpha, b)\varepsilon(\alpha, \beta)\varepsilon(a, \beta) = \varepsilon(a\alpha, b)\varepsilon(a\alpha, \beta) = \varepsilon(a\alpha, b\beta).$$

With this, it follows that $[a\alpha, b\beta] = (a\alpha)(b\beta) - \varepsilon(a\alpha, b\beta)(b\beta)(a\alpha)$ maps r to the element $a\alpha(b)\beta(r) - \varepsilon(a\alpha, b\beta)b\beta(a)\alpha(r)$. Thus, as operators on A, we have $[a\alpha, b\beta] = a\alpha(b)\beta - \varepsilon(a\alpha, b\beta)b\beta(a)\alpha$, as required. □

For the remainder of this paper, we assume that A is a color commutative, Γ-graded K-algebra so that condition (C1) holds. Furthermore, we let Δ be a color commutative Lie color subalgebra of $\mathrm{Der}^{\varepsilon} A$. Thus $\Delta = \oplus\sum_{x\in\Gamma} \Delta_x$ where $\Delta_x = \Delta \cap \mathrm{Der}^{\varepsilon}_x(A)$, and $[\Delta, \Delta] = 0$. In particular, the latter is equivalent to

C2. $\alpha\beta = \varepsilon(\alpha, \beta)\beta\alpha$ for all $\alpha, \beta \in \mathcal{H}(\Delta)$.

Note that the tensor product $A \otimes_K \Delta = A\Delta$ is Γ-graded by defining

L1. $(A\Delta)_x = \sum_{yz=x} A_y\Delta_z$ for all $x \in \Gamma$.

In view of Lemma 1.2(i), we now have a graded action of $A\Delta$ on A, where $a\alpha \in A\Delta$ acts via its corresponding operator in $\mathrm{Der}^{\varepsilon}(A)$. Moreover, Lemma 1.2(ii) motivates us to define the binary operation [,] on $A\Delta$ as the K-linear extension of

L2. $[a\alpha, b\beta] = a\alpha(b)\beta - \varepsilon(a\alpha, b\beta)b\beta(a)\alpha$ for all $a, b \in \mathcal{H}(A)$, $\alpha, \beta \in \mathcal{H}(\Delta)$.

With all of this, we obtain

Proposition 1.3. *Let A and Δ be as above. Then $A\otimes\Delta = A\Delta$ is a Lie color algebra with Γ-grading given by* (L1) *and Lie color bracket given by* (L2). *Furthermore, the action of $A\Delta$ on A yields a Lie color homomorphism $\theta\colon A\Delta \to \mathrm{Der}^{\varepsilon}(A)$.*

Proof. It is clear that $[\ ,\ ]$ is K-linear in each variable. Thus it remains to verify the color versions of skew symmetry and the Jacobi identity as given in [S, (3.3) and (3.4)]. To this end, let $a, b, c \in \mathcal{H}(A)$ and let $\alpha, \beta, \gamma \in \mathcal{H}(\Delta)$. Then

$$[a\alpha, b\beta] = -\varepsilon(a\alpha, b\beta)[b\beta, a\alpha]$$

follows immediately from (L2) and (E1). For the Jacobi identity, we note that

$$[a\alpha, [b\beta, c\gamma]] = [a\alpha, b\beta(c)\gamma - \varepsilon(b\beta, c\gamma)c\gamma(b)\beta].$$

Hence (D2) yields

$$\varepsilon(c\gamma, a\alpha)[a\alpha, [b\beta, c\gamma]] = J_1 + J_2 + J_3 - J_4 - J_5 - J_6$$

where

$$\begin{aligned}
J_1 &= \varepsilon(c\gamma, a\alpha)a\alpha(b)\beta(c)\gamma \\
J_2 &= -\varepsilon(c\gamma, a\alpha)\varepsilon(b\beta, c\gamma)\varepsilon(\alpha, c)ac(\alpha\gamma)(b)\beta \\
J_3 &= \varepsilon(c\gamma, a\alpha)\varepsilon(a\alpha, b\beta c\gamma)\varepsilon(b\beta, c\gamma)c\gamma(b)\beta(a)\alpha \\
&= \varepsilon(a\alpha, b\beta)\varepsilon(b\beta, c\gamma)c\gamma(b)\beta(a)\alpha \\
J_4 &= \varepsilon(c\gamma, a\alpha)\varepsilon(a\alpha, b\beta c\gamma)b\beta(c)\gamma(a)\alpha \\
&= \varepsilon(a\alpha, b\beta)b\beta(c)\gamma(a)\alpha \\
J_5 &= -\varepsilon(c\gamma, a\alpha)\varepsilon(\alpha, b)ab(\alpha\beta)(c)\gamma \\
J_6 &= \varepsilon(c\gamma, a\alpha)\varepsilon(b\beta, c\gamma)a\alpha(c)\gamma(b)\beta.
\end{aligned}$$

For convenience, set $f(a\alpha, b\beta, c\gamma) = J_1 + J_2 + J_3$ and note that $f(b\beta, c\gamma, a\alpha)$ is obtained from $f(a\alpha, b\beta, c\gamma)$ by a cyclic shift of the variables. Now it is easy to see that the shifted version of J_1 is equal to J_4 and that the shifted version of J_3 is J_6. Furthermore, it follows from (C1) and (C2) that the shifted version of J_2 is

$$\begin{aligned}
&-\varepsilon(a\alpha, b\beta)\varepsilon(c\gamma, a\alpha)\varepsilon(\beta, a)\cdot ba\cdot(\beta\alpha)(c)\gamma \\
&\qquad = -\varepsilon(a\alpha, b\beta)\varepsilon(c\gamma, a\alpha)\varepsilon(\beta, a)\cdot\varepsilon(b, a)ab\cdot\varepsilon(\beta, \alpha)(\alpha\beta)(c)\gamma \\
&\qquad = -\varepsilon(c\gamma, a\alpha)\varepsilon(\alpha, b)ab(\alpha\beta)(c)\gamma = J_5.
\end{aligned}$$

In other words,

$$\varepsilon(c\gamma, a\alpha)[a\alpha, [b\beta, c\gamma]] = f(a\alpha, b\beta, c\gamma) - f(b\beta, c\gamma, a\alpha)$$

and the color Jacobi identity follows by adding the three shifted versions of this expression.

Of course, it is clear from Lemma 1.2 and (L2) that θ is indeed a Lie color homomorphism. $\square$

As we observed earlier, if $a = 1$, then $\beta(a) = \beta(1) = 0$. Thus, in this special case, equation (L2) reduces to $[1\alpha, b\beta] = \alpha(b)\beta$. Furthermore, since the bicharacter does not appear here, it follows from linearity that

L3. $[1\alpha, b\beta] = \alpha(b)\beta \qquad$ for all $b \in A,\ \alpha, \beta \in \Delta$.

§2. Necessary Conditions for Simplicity

The goal of this paper is to determine when $A \otimes \Delta = A\Delta$ is a simple Lie color algebra. Recall that L is a Lie color ideal of $A\Delta$ if

I1. L is a Γ-graded K-subspace of $A\Delta$.
I2. $[A\Delta, L] = [L, A\Delta] \subseteq L$.

Furthermore, if $A\Delta$ has only the trivial Lie color ideals, namely $L = 0$ and $L = A\Delta$, then $A\Delta$ is said to be simple. Note that if the bicharacter satisfies $\varepsilon(\Gamma, \Gamma) = 1$, then $A\Delta$ is really an ordinary Lie algebra. But, in view of (I1), we see that $A\Delta$ is a simple Lie color algebra if and only if it is a graded simple Lie algebra.

We continue with the notation of the preceding section. As in [Pa], there are two obvious necessary conditions for simplicity. To start with, A must be graded Δ-simple. This means that A has no nontrivial graded Δ-stable ideals. Indeed, if I is a graded Δ-stable ideal of A, then $I \otimes \Delta = I\Delta$ is clearly a Lie color ideal of $A\Delta$. Hence, if $A\Delta$ is simple, then I must 0 or A. Note that the graded Δ-simplicity of A also forces the ring of constants A^Δ to have a rather nice structure.

Lemma 2.1. *If A is graded Δ-simple, then all elements of $\mathcal{H}(A^\Delta)$ are invertible in this subalgebra. In particular, A^Δ is a crossed product $B*\tilde{\Gamma}$ where $B = (A^\Delta)_1$ is the identity component of the subalgebra and where $\tilde{\Gamma}$ is some subgroup of Γ.*

Proof. We know that A^Δ is a graded subalgebra of A. Furthermore, if $a \in \mathcal{H}(A^\Delta)$, then $aA = Aa$ is a nonzero graded Δ-stable ideal of A, by Lemma 1.1(i), and hence $aA = Aa = A$. Thus a is invertible in A and let $b = A^{-1} \in \mathcal{H}(A)$. Finally, for any $\partial \in \mathcal{H}(\Delta)$, equation (D2) implies that $0 = \partial(1) = \partial(ba) = \partial(b)a$, so $\partial(b) = 0$ and $b \in A^\Delta$, as required. $\square$

The graded Δ-simplicity of A is also somewhat related to the parity function on Γ. Specifically, note that $\varepsilon(x,x)^2 = 1$ by (E1) and therefore $\varepsilon(x,x) = \pm 1$. Furthermore, (E2) implies that the map $\Gamma \to \{\pm 1\}$ given by $x \mapsto \varepsilon(x,x)$ is a group homomorphism. Thus the kernel of this map, $\Gamma_+ = \{x \in \Gamma \mid \varepsilon(x,x) = 1\}$, is a subgroup of Γ of index ≤ 2, and if this subgroup has index 2 then its second coset is given by $\Gamma_- = \{\, x \in \Gamma \mid \varepsilon(x,x) = -1 \,\}$. Of course, if $\operatorname{char} K = 2$, then we always have $\Gamma = \Gamma_+$. For convenience, if V is a Γ-graded, K-vector space, then we use $\mathcal{H}_+(V)$ to denote the set of homogeneous elements of V with grade in Γ_+, and we let V_+ be the K-linear span of $\mathcal{H}_+(V)$. Similarly, we let $\mathcal{H}_-(V)$ denote the set of homogeneous elements with grade in Γ_-, and V_- is its span. Clearly, $V = V_+ \oplus V_-$.

Lemma 2.2. *$AA_- = A_-A$ is a graded nil ideal of A. In particular, if A is graded Δ-simple and if $\Delta = \Delta_+$, then $A = A_+$.*

Proof. If $a \in \mathcal{H}_-(A)$, then (C1) implies that $a^2 = aa = \varepsilon(a,a)aa = -a^2$. Thus $a^2 = 0$ since $\operatorname{char} K \neq 2$ in this case. It now follows from Lemma 1.1(iii) that A_-A is a graded nil ideal and hence $A_-A \neq A$. Finally, if $\Delta = \Delta_+$, then it is clear that A_-A is Δ-stable. In particular, if A is graded Δ-simple, then $A_-A = 0$ and therefore $A = A_+$. □

A second necessary condition for the simplicity of $A\Delta$ is that this Lie color algebra act faithfully on A. Indeed, if $A\Delta$ is simple, then the kernel of the Lie color homomorphism θ of Proposition 1.3 is either 0 or $A\Delta$. But, in the latter case, $1\Delta = \Delta$ acts trivially on A, contradicting its definition as a nonzero subspace of $\operatorname{Der}^\varepsilon(A)$. Thus $A\Delta$ must act faithfully and hence so must its subspace $A^\Delta \otimes \Delta = A^\Delta\Delta$. As we see below, all that is required is this weaker assumption. Specifically, we will prove

Theorem 2.3. *Let $\varepsilon\colon \Gamma \times \Gamma \to K^\bullet$ be a bicharacter for the multiplicative group Γ. Suppose A is a color commutative Γ-graded, K-algebra and let Δ be a nonzero color commutative Γ-graded, K-vector space of color derivations of A. If $\dim_K \Delta = 1$, assume in addition that $\operatorname{char} K \neq 2$ and that $\Delta = \Delta_+$. Then $A \otimes \Delta = A\Delta$ is a simple Lie color algebra if and only if A is graded Δ-simple and $A^\Delta \otimes \Delta = A^\Delta\Delta$ acts faithfully on A.*

As is apparent, there is two missing cases here which occur when $\dim_K \Delta = 1$. In this situation, we have

Lemma 2.4. *If A is graded Δ-simple and $\dim_K \Delta = 1$, then $A^\Delta \otimes \Delta = A^\Delta\Delta$ acts faithfully on A.*

Proof. Say $\Delta = K\partial$ and consider the restriction of θ to $A^\Delta \otimes \Delta = A^\Delta\Delta$. Since θ is a graded homomorphism, the kernel of θ is generated by its homogeneous elements. But $0 \neq \partial \in \mathcal{H}(\Delta)$ and all elements of $\mathcal{H}(A^\Delta)$ are units. Thus $\ker\theta$ contains no nonzero homogeneous elements and $A^\Delta\Delta$ acts faithfully, as required. □

Thus, when Δ is 1-dimensional, the assumption that $A^\Delta\Delta$ acts faithfully can be dropped. However, now another condition comes into play.

Theorem 2.5. *Suppose A is a color commutative Γ-graded, K-algebra and let Δ be a nonzero color commutative Γ-graded, K-vector space of color derivations of A. Let $\dim_K \Delta = 1$ and assume that either $\operatorname{char} K = 2$ or $\operatorname{char} K \neq 2$ and $\Delta = \Delta_-$.*

Then $A\otimes\Delta = A\Delta$ is a simple Lie color algebra if and only if A is graded Δ-simple and $\Delta(A) = A$.

At this point, an example would be helpful. Here we use the usual bicharacter associated with superalgebras.

Example 2.6. *Let $\Gamma = \{1, g\}$ have order 2 and let ε be defined so that $\varepsilon(g, g) = -1$. Suppose F is a field extension of K and let δ be an ordinary K-derivation of F. If $A = F[t]/(t^2)$, then A is a Γ-graded, color commutative algebra with $A_1 = F$ and $A_g = F\bar{t}$, where $\bar{t}$ is the image in A of the variable t. Now let $\partial\colon A \to A$ be given by $\partial(r + s\bar{t}) = s + \delta(r)\bar{t}$ for all $r, s \in F$. Then ∂ is a color derivation of A of grade $\partial^{\#} = g$. Furthermore, if $\Delta = K\partial$, then A is graded Δ-simple, and $A\Delta$ is a simple Lie color algebra if and only if δ is onto.*

Proof. Since $\varepsilon(x, y) = 1$ if one of the group elements x or y is 1, it follows easily that A is Γ-graded and color commutative. Furthermore, it is easy to verify that ∂ is a color derivation of A of grade $\partial^{\#} = g$. Now the only proper ideal of A is $F\bar{t}$, and this ideal is not Δ-stable. Consequently, A is graded, Δ-simple. Finally, since $\partial(A) = A$ if and only if $\delta(F) = F$, the preceding theorem implies that $A\Delta$ is simple if and only if δ is onto. □

If $\delta(F) = 0$ in the above, then it is easy to see that $F\partial$ is a Lie color ideal of $A\Delta$. Indeed, this follows since $[F\partial, F\partial] = 0$ and $[F\partial, F\bar{t}\partial] = F\partial$. On the other hand, as was shown in [KN], numerous purely transcendental field extensions $F \supseteq K$ exist, in any characteristic, with $\delta(F) = F$. Note that Example 2.6 also applies when $\operatorname{char} K = 2$. Here $\Gamma = \Gamma_+$ and ∂ is an ordinary derivation of A.

We close this section by isolating some specific computations which will be required later. They, of course, apply to an arbitrary $A\Delta$.

Lemma 2.7. *Let $a, b \in \mathcal{H}(A)$ and let $\partial \in \mathcal{H}(\Delta)$.*

i. *$\varepsilon(\partial, a)[a\partial, b\partial] = \big[\partial(ab) - 2\partial(a)b\big]\partial$ if $\varepsilon(\partial, \partial) = 1$.*
ii. *$\varepsilon(\partial, a)[a\partial, b\partial] = \partial(ab)\partial$ when $\varepsilon(\partial, \partial) = -1$.*
iii. *$\partial(b^n) = n\partial(b)b^{n-1}$ if $n \geq 1$ and $\varepsilon(b, b) = 1$.*

Proof. Write $\varepsilon(\partial, a)[a\partial, b\partial] = c\partial$ for a suitable $c \in A$. Then

$$\begin{aligned} c &= \varepsilon(\partial, a)a\partial(b) - \varepsilon(\partial, a)\varepsilon(a\partial, b\partial)b\partial(a) \\ &= \varepsilon(\partial, a)a\partial(b) - \varepsilon(\partial, \partial)\varepsilon(\partial(a), b)b\partial(a) \\ &= \varepsilon(\partial, a)a\partial(b) - \varepsilon(\partial, \partial)\partial(a)b \end{aligned}$$

since (C1) implies that $\partial(a)b = \varepsilon(\partial(a), b)b\partial(a)$. Thus

$$c = \partial(ab) - \partial(a)b - \varepsilon(\partial, \partial)\partial(a)b,$$

by (D2), so parts (i) and (ii) follow.

For (iii), we proceed by induction on $n \geq 1$, and we suppose that the result holds for n. Then

$$\begin{aligned}\partial(b^{n+1}) = \partial(bb^n) &= \partial(b)b^n + \varepsilon(\partial, b)b\partial(b^n)\\ &= \partial(b)b^n + n\varepsilon(\partial, b)b\partial(b)b^{n-1} = (n+1)\partial(b)b^n\end{aligned}$$

since $\varepsilon(b, b) = 1$ and (C1) imply that $\varepsilon(\partial, b)b\partial(b) = \varepsilon(b\partial, b)b\partial(b) = \partial(b)b$. □

Of course, if $b \in \mathcal{H}_-(A)$, then we know that $b^n = 0$ for $n \geq 2$. We will also need

Lemma 2.8. *Let $a, r \in \mathcal{H}(A)$ and let $\partial_1, \partial_2 \in \mathcal{H}(\Delta)$. Then*

$$\varepsilon(a, \partial_1)[r\partial_1, a\partial_2] - [ra\partial_1, 1\partial_2] = \varepsilon(a\partial_1, \partial_2)r\partial_2(a)\partial_1 + \varepsilon(a, \partial_1)r\partial_1(a)\partial_2.$$

Proof. To start with, we have

$$\begin{aligned}\varepsilon(a, \partial_1)[r\partial_1, a\partial_2] &= \varepsilon(a, \partial_1)r\partial_1(a)\partial_2 - \varepsilon(a, \partial_1)\varepsilon(r\partial_1, a\partial_2)a\partial_2(r)\partial_1\\ &= \varepsilon(a, \partial_1)r\partial_1(a)\partial_2 - \varepsilon(ra\partial_1, \partial_2)\varepsilon(r\partial_2, a)a\partial_2(r)\partial_1.\end{aligned}$$

Furthermore,

$$\begin{aligned}-[ra\partial_1, 1\partial_2] &= \varepsilon(ra\partial_1, \partial_2)\partial_2(ra)\partial_1\\ &= \varepsilon(ra\partial_1, \partial_2)\varepsilon(\partial_2, r)r\partial_2(a)\partial_1 + \varepsilon(ra\partial_1, \partial_2)\partial_2(r)a\partial_1\\ &= \varepsilon(a\partial_1, \partial_2)r\partial_2(a)\partial_1 + \varepsilon(ra\partial_1, \partial_2)\varepsilon(r\partial_2, a)a\partial_2(r)\partial_1,\end{aligned}$$

since $\partial_2(r)a = \varepsilon(r\partial_2, a)a\partial_2(r)$ by (C1). Thus, by adding the latter two displayed equations, we obtain

$$\varepsilon(a, \partial_1)[r\partial_1, a\partial_2] - [ra\partial_1, 1\partial_2] = \varepsilon(a\partial_1, \partial_2)r\partial_2(a)\partial_1 + \varepsilon(a, \partial_1)r\partial_1(a)\partial_2,$$

as required. □

§3. Sufficient Conditions for Simplicity

In this section, we prove the main theorems. Thus, we assume throughout that Γ is a multiplicative abelian group and that $\varepsilon\colon \Gamma \times \Gamma \to K^\bullet$ is a fixed bicharacter to the field K. Furthermore, A is a color commutative, Γ-graded, associative K-algebra and Δ is a nonzero color commutative Lie color subalgebra of $\mathrm{Der}^\varepsilon(A)$. Of course, we suppose that

A1. A is graded Δ-simple
A2. $A^\Delta\Delta$ acts faithfully on A.

Finally, we let L be a nonzero Lie color ideal of $A \otimes \Delta = A\Delta$.

If V is a K-subspace of A, then the sum of all left (or right) ideals of A contained in V is the unique largest left (or right) ideal of A contained in V. In particular, the ideals I mentioned in part (i) of the following lemma always exist.

Lemma 3.1. *Let V be a Δ-stable, Γ-graded K-subspace of A.*

i. *If I is the largest left (or right) ideal of A contained in V, then I is a Δ-stable, graded two-sided ideal. In particular, if V contains a nonzero left (or right) ideal, then $V = A$.*
ii. *AV and VA are Δ-stable, graded two-sided ideals of A. Thus $V \neq 0$ implies that $AV = VA = A$ and hence that $AV_+ = V_+A = A$.*

Proof. (i) Assume that I is the unique largest left ideal of A contained in V. For each $x \in \Gamma$, let $\pi_x: A \to A_x$ denote the natural projection, and set $J = \oplus \sum_{x\in\Gamma} \pi_x(I)$. Since $A_x\pi_y(I) \subseteq \pi_{xy}(A_xI) \subseteq \pi_{xy}(I)$ for all $x, y \in \Gamma$, it follows that $A_xJ \subseteq J$ and hence that J is a left ideal of A. But $I \subseteq J \subseteq V$, since V is graded, and hence $I = J$ is a graded left ideal.

Now let $\partial \in \mathcal{H}(\Delta)$. Then $I + \partial(I) \subseteq V$ and $I + \partial(I)$ is a left ideal of A since, for all $a \in \mathcal{H}(A)$ and $r \in \mathcal{H}(I)$, we have $\varepsilon(\partial, a)a\partial(r) = -\partial(a)r + \partial(ar) \in I + \partial(I)$. Thus $I + \partial(I) \subseteq I$ by the maximal nature of I, and hence I is Δ-stable. Note that $I \triangleleft A$ by Lemma 1.1(i). Finally, if V contains a nonzero left (or right) ideal, then I is a nonzero graded, Δ-stable two-sided ideal of A. But A is graded Δ-simple, so $I = A$ and therefore $V = A$.

(ii) Certainly AV is a graded left ideal of A and hence $AV = VA \triangleleft A$ by Lemma 1.1(i). Furthermore, AV is Δ-stable since both A and V are Δ-stable and graded. In particular, if $V \neq 0$, then the graded Δ-simplicity of A implies that $AV = A$. Finally, note that $A = AV = AV_+ + AV_-$. Furthermore, by Lemma 1.1(iii), AV_- is a nil ideal of A and hence it is contained in the Jacobson radical of the algebra. With this, the previous sum implies that $A = AV_+ = V_+A$. □

Such subspaces V are of interest because of

Lemma 3.2. *Let $\partial_1, \partial_2, \dots, \partial_n$ be K-linearly independent elements of $\mathcal{H}(\Delta)$ and let V be the set of all $a \in A$ such that there exists $\alpha = a_1\partial_1 + a_2\partial_2 + \cdots + a_n\partial_n \in L$ with $a = a_n$. Then V is a graded Δ-stable, K-subspace of A.*

Proof. Since L is a K-subspace of $A\Delta$, it is clear that V is a K-subspace of A. Furthermore, if $x \in \Gamma$, then since L is a graded subspace of $A\Delta$, it follows that $\sum_{i=1}^n (a_i)_{x\partial_i^{-1}}\partial_i = \alpha_x \in L$. Thus $V_{x\partial_n^{-1}} \subseteq V$ and V is a graded subspace of A since $x \in \Gamma$ is arbitrary. Finally, if $\partial \in \Delta$, then $1\partial \in A\Delta$ and equation (L3) implies that $\partial(a_1)\partial_1 + \partial(a_2)\partial_2 + \cdots + \partial(a_n)\partial_n = [1\partial, \alpha] \in L$ since $L \triangleleft A\Delta$. Thus $\partial(a) = \partial(a_n) \in V$ and V is Δ-stable. □

It is now a simple matter to prove

Proposition 3.3. *$A\Delta$ acts faithfully on A.*

Proof. Let $L \triangleleft A\Delta$ be the kernel of the action of $A\Delta$ on A and assume, by way of contradiction, that $L \neq 0$. Then we can let $n \geq 1$ be the minimal homogeneous support size of a nonzero element of L. In other words, there exist linearly independent homogeneous elements $\partial_1, \partial_2, \dots, \partial_n$ in Δ with $L \cap \sum_{i=1}^n A\partial_i \neq 0$ and such that all nonzero elements in this intersection have all their A-coefficients nonzero. Let V be defined as in the previous lemma. Then V is a nonzero graded Δ-stable, K-subspace of A. Furthermore, the nature of the action of $A\Delta$ on A implies that $AL \subseteq L$ and hence $AV \subseteq V$. Thus $V \triangleleft A$ by Lemma 1.1(i) and, since A is graded Δ-simple, we have $V = A$. In particular, $1 \in V$ and we can find $\alpha = a_1\partial_1 + a_2\partial_2 + \cdots + a_n\partial_n \in L$ with $a_n = 1$. If $\partial \in \Delta$ is arbitrary, then equation (L3) implies that $\partial(a_1)\partial_1 + \partial(a_2)\partial_2 + \cdots + \partial(a_{n-1})\partial_{n-1} = [1\partial, \alpha] \in L$ since $\partial(a_n) = \partial(1) = 0$. Thus the minimality of n implies that $\partial(a_i) = 0$ for all i and all $\partial \in \Delta$. In other words, $a_i \in A^\Delta$ for all i, and $\alpha \in A^\Delta\Delta$. But, by assumption, $A^\Delta\Delta$ acts faithfully on A, so we have the required contradiction. We conclude, therefore, that $L = 0$ and hence that $A\Delta$ acts faithfully on A. □

Recall that L is a fixed nonzero Lie color ideal of $A\Delta$. As a consequence of the above, we obtain

Lemma 3.4. *If $\partial \in \mathcal{H}(\Delta)$, then $L \cap A\partial \neq 0$.*

Proof. Since $L \neq 0$, we can choose $n \geq 1$ minimal so that there exist linearly independent homogeneous elements $\partial_1, \partial_2, \dots, \partial_n \in \Delta$ with $\partial = \partial_1$ and $L \cap \sum_{i=1}^n A\partial_i \neq 0$. The goal is to show that $n = 1$. Thus suppose that $n \geq 2$ and let V be defined as in Lemma 3.2. Then V is a graded Δ-stable, K-subspace of A and the minimality of n implies that $V \neq 0$.

Now let $\alpha = \sum_{i=1}^n a_i\partial_i$ be a nonzero homogeneous element contained in the nonzero graded vector space $L \cap \sum_{i=1}^n A\partial_i$. If $r \in \mathcal{H}(A)$, then $[\alpha, r\partial_1] \in L$ and it is clear from equation (L2) that this element has support in $\{\partial_1, \partial_2, \dots, \partial_n\}$. Thus we can write $[\alpha, r\partial_1] = \sum_{i=1}^n c_i(r)\partial_i$. The most complicated coefficient here is that of ∂_1. Specifically, $c_1(r) = -\varepsilon(a_1\partial_1, r\partial_1)r\partial_1(a_1) + \sum_{i=1}^n a_i\partial_i(r)$ and consequently $c_1(\mathcal{H}(A)) \neq 0$. Indeed, if $c_1(\mathcal{H}(A)) = 0$, then $r = 1$ implies that $\partial_1(a_1) = 0$ and hence $0 = c_1(r) = \sum_{i=1}^n a_i\partial_i(r) = \alpha(r)$ for all $r \in \mathcal{H}(A)$. In other words, α acts trivially on $\mathcal{H}(A)$, so linearity implies that it acts trivially on all of A. Thus $\alpha = 0$ by the previous proposition, a contradiction.

We have therefore shown that $[\alpha, r\partial_1]$ is not identically 0, so the minimal nature of $n \geq 2$ implies that $c_n(\mathcal{H}(A)) \neq 0$. Here $c_n(r) = -\varepsilon(a_n\partial_n, r\partial_1)r\partial_1(a_n)$, so $\partial_1(a_n) \neq 0$ and V contains $\mathcal{H}(A)\partial_1(a_n)$. Thus V contains the nonzero left ideal $A\partial_1(a_n)$, and it follows from Lemma 3.1(i) that $V = A$. Thus there exists $\beta \in L \cap \sum_{i=1}^n A\partial_i$ with $\beta = \sum_{i=1}^n b_i\partial_i$ and $b_n = 1$. Furthermore, by taking α to be the $\partial_n^\#$ component of β, we see that we could have chosen α with $a_n = 1$. But then $\partial_1(a_n) = 0$, a contradiction. Thus $n = 1$ and $0 \neq L \cap A\partial_1 = L \cap A\partial$, as required. □

Next, we see that only a small part of Δ has to be studied in detail. Indeed, we prove

Lemma 3.5. *If $L \supseteq A\partial$ for some $\partial \in \mathcal{H}(\Delta)$, then $L = A\Delta$.*

Proof. Fix $b \in \mathcal{H}(A)$ with $\partial(b) \neq 0$, let $\partial' \in \mathcal{H}(\Delta)$ be arbitrary and define $V' = \{ a \in A \mid a\partial' \in L \}$. As usual, V' is a Γ-graded, Δ-stable K-subspace of A. Since $L \supseteq A\partial$, it follows that

$$a\partial(b)\partial' = [a\partial, b\partial'] + \varepsilon(a\partial, b\partial')b\partial'(a)\partial \in L$$

for all $a \in \mathcal{H}(A)$. Hence V' contains the nonzero left ideal $A\partial(b)$, and Lemma 3.1(i) implies that $V' = A$. In other words, L contains $A\partial'$ for all $\partial' \in \mathcal{H}(\Delta)$, so $L = A\Delta$ as required. □

Lemma 3.6. *Let ∂_1 and ∂_2 be K-linearly independent elements of $\mathcal{H}(\Delta)$.*

i. *There exist $a, b \in \mathcal{H}(A)$ with*

$$\varepsilon(\partial_2, a)\partial_1(a)\partial_2(b) + \varepsilon(\partial_1, a\partial_2)\partial_2(a)\partial_1(b) \neq 0.$$

ii. *The composition $\partial_1\partial_2$ is not zero.*

Proof. (i) Choose $a \in \mathcal{H}(A)$ with $\partial_1(a) \neq 0$. Then

$$\alpha = \varepsilon(\partial_2, a)\partial_1(a)\partial_2 + \varepsilon(\partial_1, a\partial_2)\partial_2(a)\partial_1$$

is a nonzero element of $A\Delta$. Thus, since $A\Delta$ acts faithfully on A by Proposition 3.3, there exists $b \in \mathcal{H}(A)$ with $\varepsilon(\partial_2, a)\partial_1(a)\partial_2(b) + \varepsilon(\partial_1, a\partial_2)\partial_2(a)\partial_1(b) = \alpha(b) \neq 0$.

(ii) Let $a, b \in \mathcal{H}(A)$ be given by part (i) above. Then

$$\begin{aligned}\partial_1\partial_2(ab) &= \partial_1\big(\partial_2(a)b + \varepsilon(\partial_2, a)a\partial_2(b)\big)\\ &= \partial_1\partial_2(a)b + \varepsilon(\partial_1\partial_2, a)a\partial_1\partial_2(b) + c\end{aligned}$$

where $c = \varepsilon(\partial_2, a)\partial_1(a)\partial_2(b) + \varepsilon(\partial_1, a\partial_2)\partial_2(a)\partial_1(b) \neq 0$. It follows that $\partial_1\partial_2(r) \neq 0$ for $r = a$ or b or ab, and hence $\partial_1\partial_2 \neq 0$. □

We can now obtain the following crucial fact.

Lemma 3.7. *Let* $\operatorname{char} K \neq 2$. *If $\partial \in \mathcal{H}_+(\Delta)$, then $L \supseteq A\partial$.*

Proof. Let $V = \{ a \in A \mid a\partial \in L \}$. Then, by Lemma 3.2, V is a graded Δ-stable, K-subspace of A. Of course, Lemma 3.4 implies that $V \neq 0$. If $v \in \mathcal{H}(V)$ and $r \in \mathcal{H}(A)$, then $[v\partial, r\partial] \in L$ and, since $\partial \in \mathcal{H}_+(\Delta)$, Lemma 2.7(i) yields

$$\partial(vr) - 2\partial(v)r \in V \qquad \text{for all } v \in \mathcal{H}(V),\ r \in \mathcal{H}(A). \tag{1}$$

Suppose in addition that $v \in \mathcal{H}_+(V)$. Then $\partial(v^2) = 2\partial(v)v$ by Lemma 2.7(iii), so replacing r by vr in (1) implies that the expression

$$\partial(v^2r) - 2\partial(v)vr = \partial(v^2r) - \partial(v^2)r = \varepsilon(\partial, v^2)v^2\partial(r)$$

is contained in V. Consequently, we have

$$v^2\partial(r) \in V \qquad \text{for all } v \in \mathcal{H}_+(V),\ r \in \mathcal{H}(A). \tag{2}$$

Fix $a \in \mathcal{H}_+(V)$ and $b \in \mathcal{H}_+(A)$. Then, by equation (2), we have $a^2\partial(b) \in V$. Also $\partial(b^2/2) = \partial(b)b$ by Lemma 2.7(iii), so $a^2\partial(b)b \in V$. Using $v = a^2\partial(b)$ and $r = bs$ in (1), we get

$$\partial(a^2\partial(b)bs) - 2\partial(a^2\partial(b))bs \in V \qquad \text{for all } s \in \mathcal{H}(A). \tag{3}$$

Similarly, if we use $v = a^2\partial(b)b$ and $r = s$ in (1), we get

$$\partial(a^2\partial(b)bs) - 2\partial(a^2\partial(b)b)s \in V \qquad \text{for all } s \in \mathcal{H}(A). \tag{4}$$

Thus, by subtracting expression (4) from expression (3), we obtain

$$2\varepsilon(\partial, a^2\partial(b))a^2\partial(b)^2 s = 2\big[\partial(a^2\partial(b)b) - \partial(a^2\partial(b))b\big]s \in V$$

for all $s \in \mathcal{H}(A)$, and therefore $V \supseteq a^2\partial(b)^2 A$.

It remains to show that we can choose suitable a and b with $a^2\partial(b)^2 \neq 0$. To this end, note that $V \neq 0$ so $V_+A = A$ by Lemma 3.1(ii). Similarly, $\partial(A) \neq 0$ and this is a graded Δ-stable, K-subspace of A by the color commutativity rule (C2). Thus, since $\partial^\# \in \Gamma_+$, we have $\partial(A)_+ = \partial(A_+)$ and hence $\partial(A_+)A = A$ by Lemma 3.1(ii) again. Consequently, $A = V_+A = V_+\partial(A_+)A$, and it follows from Lemma 1.1(iii) that not all homogeneous generators of this ideal can be nilpotent. In particular, there exist $a \in \mathcal{H}_+(V)$ and $b \in \mathcal{H}_+(A)$ with $a\partial(b)$ not nilpotent. Hence, by (C1), $a^2\partial(b)^2 \neq 0$, and therefore V contains the nonzero right ideal $a^2\partial(b)^2 A$. We conclude from Lemma 3.1(i) that $V = A$, as required. □

The rest of the argument is somewhat more technical. To start with, we need

Lemma 3.8. *Let $\partial \in \mathcal{H}(\Delta)$ and assume that either $\operatorname{char} K = 2$ or $\partial \in \mathcal{H}_-(\Delta)$. Then $L \cap A\partial \supseteq \partial(A)\partial = [A\partial, A\partial]$. In particular, if $\dim_K \Delta = 1$, then $A\Delta$ is simple if and only if $\Delta(A) = A$.*

Proof. Say $L \cap A\partial = V\partial$ so that, as usual, V is a nonzero graded Δ-stable, K-subspace of A with $VA = A$ by Lemma 3.1(ii). Now if $a, b \in \mathcal{H}(A)$, then Lemma 2.7(i)(ii) implies that $[a\partial, b\partial] = \partial(ab)\partial$ under either hypothesis, and consequently $[A\partial, A\partial] = \partial(A)\partial$. Furthermore, if $a \in \mathcal{H}(V)$ and $b \in \mathcal{H}(A)$, then $[a\partial, b\partial] \in L$, so $V \supseteq \partial(VA) = \partial(A)$ and hence $L \supseteq \partial(A)\partial = [A\partial, A\partial]$.

Finally, suppose $\Delta = K\partial$ is 1-dimensional so that $\Delta(A) = \partial(A)$. If $\partial(A) = A$, then the above implies that $L \supseteq A\partial = A\Delta$, and hence $A\Delta$ is simple. On the other hand, if $\partial(A)$ is properly smaller than A, then $[A\partial, A\partial]$ is a nonzero Lie color ideal of $A\partial$ properly smaller than $A\partial$, and therefore $A\partial = A\Delta$ is not simple. □

Finally, we use some bicharacter computations to prove

Lemma 3.9. *Let* $\partial_1, \partial_2 \in \mathcal{H}(\Delta)$ *be* K*-linearly independent, and assume that either* $\operatorname{char} K = 2$ *or* $\partial_1, \partial_2 \in \mathcal{H}_-(\Delta)$*. Then* $L \supseteq A\partial_1 + A\partial_2$*.*

Proof. Write $M = A\partial_1 + A\partial_2$ so that M is a Lie color subalgebra of $A\Delta$. We proceed in a series of two steps.

Step 1. $L \supseteq [M, M]$.

Proof. Let V_2 be the set of ∂_2-coefficients in $L \cap M = L \cap (A\partial_1 + A\partial_2)$. By Lemma 2.7(ii), there exists an element $a \in \mathcal{H}(A)$ with $\partial_1\partial_2(a) \neq 0$ and we set $b = \partial_2(a)$. Then $b\partial_2 \in L \cap A\partial_2$ by the previous lemma, so for all $r \in \mathcal{H}(A)$ we have

$$r\partial_1(b)\partial_2 - \varepsilon(r\partial_1, b\partial_2)b\partial_2(r)\partial_1 = [r\partial_1, b\partial_2] \in L.$$

In other words, V_2 contains the left ideal $A\partial_1(b)$ and note that $\partial_1(b) = \partial_1\partial_2(a) \neq 0$. Lemmas 3.2 and 3.1(i) now imply that $V_2 = A$, and hence $A\partial_1 + L \supseteq A\partial_2$. But then $A\partial_1 + (M \cap L) = M$, so $M/(M \cap L) \cong A\partial_1/(A\partial_1 \cap L)$ and the latter Lie color algebra is abelian by Lemma 3.8. □

Step 2. $L \supseteq M$.

Proof. Let $a, b, r \in \mathcal{H}(A)$. Since $L \supseteq [M, M]$, we conclude from Lemma 2.8 that

$$\varepsilon(a\partial_1, \partial_2)r\partial_2(a)\partial_1 + \varepsilon(a, \partial_1)r\partial_1(a)\partial_2 \in L \qquad \text{for all } a, r \in \mathcal{H}(A). \tag{5}$$

Furthermore, replacing r by $r\partial_1(b)$ in (5) yields

$$\varepsilon(a\partial_1, \partial_2)r\partial_1(b)\partial_2(a)\partial_1 + \varepsilon(a, \partial_1)r\partial_1(b)\partial_1(a)\partial_2 \in L \qquad \text{for all } a, b, r \in \mathcal{H}(A) \tag{6}$$

and, if we interchange the order of the products $\partial_1(b)\partial_2(a)$ and $\partial_1(b)\partial_1(a)$ in the above, using (C1), we obtain

$$\lambda_1 r\partial_2(a)\partial_1(b)\partial_1 + \lambda_2 r\partial_1(a)\partial_1(b)\partial_2 \in L \qquad \text{for all } a, b, r \in \mathcal{H}(A) \tag{7}$$

where

$$\lambda_1 = \varepsilon(a\partial_1, \partial_2)\varepsilon(b\partial_1, a\partial_2) \qquad \text{and} \qquad \lambda_2 = \varepsilon(a, \partial_1)\varepsilon(b\partial_1, a\partial_1).$$

Of course, if we interchange a and b in (6), we get

$$\lambda_3 r\partial_1(a)\partial_2(b)\partial_1 + \lambda_4 r\partial_1(a)\partial_1(b)\partial_2 \in L \qquad \text{for all } a, b, r \in \mathcal{H}(A) \tag{8}$$

where

$$\lambda_3 = \varepsilon(b\partial_1, \partial_2) \qquad \text{and} \qquad \lambda_4 = \varepsilon(b, \partial_1).$$

Finally, set $\mu = \varepsilon(a\partial_1\partial_2, b)\varepsilon(\partial_2, a\partial_1)$, multiply expression (7) by $\mu\lambda_4$ and subtract $\mu\lambda_2$ times expression (8). This yields

$$r\big[\eta_1\partial_2(a)\partial_1(b) - \eta_2\partial_1(a)\partial_2(b)\big]\partial_1 \in L$$

where

$$\begin{aligned}\eta_1 &= \mu\lambda_1\lambda_4\\ &= \varepsilon(a\partial_1\partial_2, b)\varepsilon(\partial_2, a\partial_1)\varepsilon(a\partial_1, \partial_2)\varepsilon(b\partial_1, a\partial_2)\varepsilon(b, \partial_1)\\ &= \varepsilon(\partial_1, a\partial_2)\end{aligned}$$

and

$$\begin{aligned}\eta_2 &= \mu\lambda_2\lambda_3\\ &= \varepsilon(a\partial_1\partial_2, b)\varepsilon(\partial_2, a\partial_1)\varepsilon(a, \partial_1)\varepsilon(b\partial_1, a\partial_1)\varepsilon(b\partial_1, \partial_2)\\ &= \varepsilon(\partial_2, a)\varepsilon(\partial_1, \partial_1) = -\varepsilon(\partial_2, a),\end{aligned}$$

since $\varepsilon(\partial_1, \partial_1) = -1$ when either $\operatorname{char} K = 2$ or $\partial_1 \in \mathcal{H}_-(\Delta)$.

We have therefore shown that $L \cap A\partial_1 \supseteq I\partial_1$ where I is the principal left ideal Ac with $c = \varepsilon(\partial_2, a)\partial_1(a)\partial_2(b) + \varepsilon(\partial_1, a\partial_2)\partial_2(a)\partial_1(b)$. But Lemma 3.6(i) guarantees the existence of homogeneous elements a and b with $c \neq 0$. Thus we conclude from Lemmas 3.2 and 3.1(i) that $L \supseteq A\partial_1$ and, by symmetry, it follows that $L \supseteq A\partial_2$. □

There is presumably some natural reason why the element c above is so intimately related, as in Lemma 3.6(ii), to $\partial_1\partial_2(ab)$. However, we will not pursue this here. It is now a simple matter to offer the

Proof of the Main Theorems. We know that if $A\Delta$ is color simple, then A is graded Δ-simple and $A^\Delta\Delta$ acts faithfully on A. Furthermore, if $\dim_K \Delta = 1$ and if either $\operatorname{char} K = 2$ or $\operatorname{char} K \neq 2$ and $\Delta = \Delta_-$, then it follows from Lemma 3.8 that $\Delta(A) = A$. With this, the necessity aspects of Theorems 2.3 and 2.5 are proved.

For the sufficiency part of these two theorems, assume that A is graded Δ-simple and that $A^\Delta\Delta$ acts faithfully on A if $\dim_K \Delta \geq 2$. Then, by Lemma 2.4, $A^\Delta\Delta$ acts faithfully on A in all situations. Now let L be a nonzero Lie color ideal of $A\Delta$. If $\operatorname{char} K \neq 2$ and $\Delta_+ \neq 0$, then Lemma 3.7 implies that $L \supseteq A\partial$ for some $\partial \in \mathcal{H}(\Delta)$. On the other hand, if either $\operatorname{char} K = 2$ and $\dim_K \Delta \geq 2$ or $\operatorname{char} K \neq 2$ and $\dim_K \Delta_- \geq 2$, then Lemma 3.9 implies that $L \supseteq A\partial$ for some $\partial \in \mathcal{H}(\Delta)$. Thus, in either case, we conclude from Lemma 3.5 that $L = A\Delta$ and hence that $A\Delta$ is simple. Finally, suppose that $\dim_K \Delta = 1$ and that either $\operatorname{char} K = 2$ or $\operatorname{char} K \neq 2$ and $\Delta = \Delta_-$. Then Theorem 2.5 follows from Lemma 3.8 and the additional hypothesis that $\Delta(A) = A$. □

§4. Examples and Discoloration

We begin with a canonical example, intimately related to the ordinary Witt Lie algebra. As usual, we let $\varepsilon\colon \Gamma\times\Gamma\to K^\bullet$ be a fixed bicharacter. Furthermore, in the following, we use $U(\mathfrak{g})$ to denote the universal enveloping algebra of the Lie color algebra $\mathfrak{g}$ when $\operatorname{char} K = 0$ or the restricted enveloping algebra of the restricted Lie color algebra $\mathfrak{g}$ when $\operatorname{char} K = p > 0$. See [B] for the basic properties of these associative algebras.

Example 4.1. *Let $V = \sum_{x\in\Gamma} V_x$ be a nonzero Γ-graded K-vector space, and view V as a Lie color algebra by defining $[V,V]=0$. Furthermore, when $\operatorname{char} K = p > 0$, we suppose that V is restricted with trivial pth power map. If $A = U(V)$, then A is a Γ-graded, color commutative K-algebra. Now let $\Delta = \sum_{x\in\Gamma}\Delta_x$, where $\Delta_{x^{-1}} = \operatorname{Hom}_K(V_x, K)$, and view Δ as a Lie color algebra by defining $[\Delta,\Delta]=0$. Then Δ acts naturally and faithfully as color derivations on A, and with respect to this action, $A\otimes\Delta = A\Delta$ is a simple Lie color algebra unless $\dim_K V = 1$ and either $\operatorname{char} K = 2$ or $\operatorname{char} K \neq 2$ and $V = V_-$.*

Proof. Let $\mathfrak{g} = Z\oplus V\oplus\Delta$, where $0\neq Z = Kz$ and where z has grade $1\in\Gamma$. Then $\mathfrak{g}$ is a Γ-graded K-vector space, and we define $[\ ,\ ] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ in such a way that the only possible nonzero terms, involving the homogeneous generators of $\mathfrak{g}$, are given by

$$[\lambda, v] = \lambda(v)z,\quad [v,\lambda] = \varepsilon(x,x)\lambda(v)z \qquad \text{for all } x\in\Gamma,\ v\in V_x,\ \lambda\in\Delta_{x^{-1}}.$$

Since $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]=0$, it is clear that, with this definition, $\mathfrak{g}$ is now a Lie color algebra. Furthermore, if $\operatorname{char} K = p > 0$, we let $\mathfrak{g}$ be restricted by defining $\mathfrak{g}_+^{[p]} = 0$.

Next, via the adjoint map, we know that $\mathfrak{g}$ acts as color derivations on $U(\mathfrak{g})$, and in particular, $\Delta\subseteq\mathfrak{g}$ acts as color derivations on $B = U(Z\oplus V)$. Furthermore, z is central in B and there is a natural graded algebra epimorphism $\phi\colon B = U(Z\oplus V)\to A = U(V)$ determined by $\phi(z)=1$. Thus, since $\Delta(z)=0$, ϕ gives rise to a derivation action of Δ on A. Indeed, this action is faithful, since Δ is already faithful on $\phi(Z\oplus V) = K\oplus V\subseteq A$. It now makes sense to form the Lie color algebra $A\otimes\Delta = A\Delta$, and we will use the main theorems here to determine when $A\Delta$ is simple.

To start with, any element $a\in A$ can be written as

$$a = \sum_\alpha k_\alpha t_1^{\alpha_1} t_2^{\alpha_2}\cdots t_n^{\alpha_n},$$

a PBW polynomial in the variables $t_1, t_2, \dots, t_n$. Here $\{\, t_1, t_2, \dots, t_n \,\}$ is a linearly independent subset of $\mathcal{H}(V)$ and

$$\alpha_i \in \begin{cases} \{\,0,1\,\}, & \text{if } t_i\in\mathcal{H}_-(V) \\ \{\,0,1,2,\dots\,\}, & \text{if } t_i\in\mathcal{H}_+(V) \text{ and } \operatorname{char} K = 0 \\ \{\,0,1,\dots,p-1\,\}, & \text{if } t_i\in\mathcal{H}_+(V) \text{ and } \operatorname{char} K = p > 0. \end{cases}$$

Of course, this polynomial expression for a is unique once $t_1, t_2, \ldots, t_n$ is chosen, and $\deg \alpha = \sum_{i=1}^{n} \alpha_i$ is well-defined, independent of this choice. Note that, if $t_1, t_2, \ldots, t_n$ are given, then the definition of Δ implies that the "partial derivatives" $\partial_1, \partial_2, \ldots, \partial_n \in \mathcal{H}(\Delta)$ exist with $\partial_i^{\#} = (t_i^{\#})^{-1}$ and with $\partial_i(t_j) = \delta_{i,j}$, the Kronecker delta.

Now suppose that the element a above is contained in A^{Δ}. Then for all i,

$$0 = \partial_i(a) = \sum_{\alpha} k_\alpha \varepsilon(\partial_i, t_1^{\alpha_1} \cdots t_{i-1}^{\alpha_{i-1}}) \alpha_i t_1^{\alpha_1} \cdots t_i^{\alpha_i - 1} \cdots t_n^{\alpha_n}$$

by (D2) and Lemma 2.7(iii). In particular, if $k_\alpha \neq 0$, then the uniqueness aspect of this expression implies that $\alpha_i = 0$, and consequently $a = k_0 \in K$. Thus $A^{\Delta} = K$ and $A^{\Delta} \otimes \Delta = K \otimes \Delta = \Delta$ acts faithfully on A. Furthermore, let I be a nonzero graded Δ-stable ideal of A and let $0 \neq a \in I$ be an element of minimal degree. Since $\deg \partial(a) < \deg a$ for all $\partial \in \mathcal{H}(\Delta)$, it follows that $\partial(a) = 0$ and hence that $a \in A^{\Delta} = K$. But this implies that $I = A$, so A is graded Δ-simple and the result follows from Theorems 2.3 and 2.5. $\square$

The above construction, when $A\Delta$ is not simple, yields a familiar example. To start with, we have $\dim_K V = 1$, so say $V = Kt$. Thus, since either $\operatorname{char} K = 2$ or $\operatorname{char} K \neq 2$ and $t \in \mathcal{H}_-(V)$, it follows that $A = K \oplus Kt$ with $t^2 = 0$. Furthermore, by definition of the action, we have $\Delta = K\partial$ with $\partial(r + st) = s$ for all $r, s \in K$. In particular, this is precisely Example 2.6 with $F = K$ and $\delta = 0$.

Of course, starting with an arbitrary A and Δ, we can obtain additional Lie structures by considering group rings and formal exponentials as in [Pa, §1]. Since the arguments for this are fairly routine, and since these new examples may not be particularly interesting, we prefer to take a somewhat different tack. Specifically, we consider what the discoloration functor has to say about Witt-type Lie color algebras.

Let Γ and K be given, and recall that a 2-cocycle is a map $\tau: \Gamma \times \Gamma \to K^{\bullet}$ satisfying

H1. $\tau(x, yz)\tau(y, z) = \tau(x, y)\tau(xy, z)$ for all $x, y, z \in \Gamma$.

Furthermore, using τ, we can construct the twisted group algebra $K^\tau[\Gamma]$ of Γ over K. The latter is, of course, the K-algebra with basis $\overline{\Gamma} = \{\, \overline{x} \mid x \in \Gamma \,\}$ and with multiplication defined distributively by

H2. $\overline{x}\,\overline{y} = \tau(x, y)\overline{xy}$ for all $x, y \in \Gamma$.

Note that (H1) is precisely equivalent to the associativity of multiplication, and hence $K^\tau[\Gamma]$ is an associative K-algebra which is Γ-graded if we set $K^\tau[\Gamma]_x = K\overline{x}$ for all $x \in \Gamma$. Of course, if τ is trivial, that is if $\tau(\Gamma, \Gamma) = 1$, then $K^\tau[\Gamma] = K[\Gamma]$ is the ordinary group algebra of Γ.

Lemma 4.2. *Let $K^\tau[\Gamma]$ be given as above and define $\mu_\tau(x,y) = \tau(x,y)/\tau(y,x)$ for all $x, y \in \Gamma$. Then*

i. $\overline{x}\,\overline{y} = \mu_\tau(x,y)\overline{y}\,\overline{x}$ *for all $x, y \in \Gamma$.*
ii. $\mu_\tau \colon \Gamma \times \Gamma \to K^\bullet$ *is a bicharacter with $\mu_\tau(x,x) = 1$ for all $x \in \Gamma$.*

Proof. (i) This follows from (H2) since

$$\overline{x}\,\overline{y} = \tau(x,y)\overline{xy} = \tau(x,y)\overline{yx} = \tau(x,y)/\tau(y,x)\,\overline{y}\,\overline{x}.$$

(ii) Certainly $\mu_\tau(x,y)\mu_\tau(y,x) = 1$ and $\mu_\tau(x,x) = 1$. Moreover, if $x, y, z \in \Gamma$, then $\overline{yz} = k\overline{y}\,\overline{z}$ for some $k \in K^\bullet$ and hence

$$\begin{aligned}\mu_\tau(x,yz)\overline{yz}\,\overline{x} &= \overline{x}\,\overline{yz} = k\overline{x}\,\overline{y}\,\overline{z} = k\mu_\tau(x,y)\overline{y}\,\overline{x}\,\overline{z}\\ &= k\mu_\tau(x,y)\mu_\tau(x,z)\overline{y}\,\overline{z}\,\overline{x} = \mu_\tau(x,y)\mu_\tau(x,z)\overline{yz}\,\overline{x}.\end{aligned}$$

Thus $\mu_\tau(x,yz) = \mu_\tau(x,y)\mu_\tau(x,z)$, as required. □

If $V = \oplus\sum_{x\in\Gamma} V_x$ and $W = \oplus\sum_{x\in\Gamma} W_x$ are Γ-graded K-vector spaces, then we define the diagonal product $V \boxtimes W$ to be the *diagonal subspace* of $V \otimes W$ given by $V \boxtimes W = \oplus\sum_{x\in\Gamma} V_x \otimes W_x$. Of course $V \boxtimes W$ is graded with $(V \boxtimes W)_x = V_x \otimes W_x$. Furthermore, if A and B are Γ-graded K-algebras, then $A \boxtimes B$ is a subalgebra of $A \otimes B$, and it is Γ-graded with $(A \boxtimes B)_x = A_x \otimes B_x$. Since the tensor product is associative and since $(U \boxtimes V) \boxtimes W$ is the *diagonal subspace* of $(U \otimes V) \otimes W$, it follows that the diagonal product is also associative. Note that $A \boxtimes K[\Gamma] \cong A$ as Γ-graded algebras. Furthermore, if τ is a 2-cocycle, then so is $\tau^{-1} = 1/\tau$, and clearly we have $\mu_{\tau^{-1}} = \mu_\tau^{-1}$ and $K^\tau[\Gamma] \boxtimes K^{\tau^{-1}}[\Gamma] \cong K[\Gamma]$.

In the following, we will be concerned with more that one bicharacter of Γ. Thus, to be precise, we will say, for example, that an algebra is ε-commutative if it is color commutative with respect to the bicharacter ε.

Lemma 4.3. *Let ε be a bicharacter of Γ and let τ be a 2-cycle. Then $\varepsilon' = \varepsilon\mu_\tau$ is also a bicharacter of Γ and the operator T on the set of Γ-graded associative K-algebras given by $T(A) = A \boxtimes K^\tau[\Gamma]$ yields a one-to-one correspondence from the isomorphism classes of ε-commutative algebras to the isomorphism classes of ε'-commutative algebras. Here T^{-1} is given by $T^{-1}(A') = A' \boxtimes K^{\tau^{-1}}[\Gamma]$.*

Proof. If S is defined by $S(A') = A' \boxtimes K^{\tau^{-1}}[\Gamma]$, then

$$\begin{aligned}S(T(A)) &= (A \boxtimes K^\tau[\Gamma]) \boxtimes K^{\tau^{-1}}[\Gamma]\\ &= A \boxtimes (K^\tau[\Gamma] \boxtimes K^{\tau^{-1}}[\Gamma]) \cong A \boxtimes K[\Gamma] \cong A.\end{aligned}$$

Thus ST is the identity on isomorphism classes of algebras, and similarly so is TS. In other words, $S = T^{-1}$ and T is a one-to-one correspondence. Finally, the comment on ε- and ε'-commutative algebras follows immediately from $\varepsilon' = \varepsilon\mu_\tau$ and Lemma 4.2(i). $\square$

Next, we turn to derivations. Let A be a Γ-graded algebra and let ∂ be a homogeneous ε-derivation of A of grade $\partial^\# = z$. Then $\partial \otimes \overline{z}$ acts on $A \boxtimes K^\tau[\Gamma]$ by

H3. $(\partial \otimes \overline{z})\colon a \otimes \overline{x} \mapsto \partial(a) \otimes \overline{z}\,\overline{x} \qquad$ for all $x \in \Gamma,\ a \in A_x$,

and it is easy to see that $\partial \otimes \overline{z}$ is an ε'-derivation of $A \boxtimes K^\tau[\Gamma]$. Indeed, let $x, y \in \Gamma$ and let $a \in A_x$, $b \in A_y$. Since

$$\varepsilon(z,x)\overline{z}\,\overline{x} = \varepsilon(z,x)\mu_\tau(z,x)\overline{z}\,\overline{x} = \varepsilon'(z,x)\overline{x}\,\overline{z},$$

by Lemma 4.2(i), it follows that

$$\begin{aligned}(\partial \otimes \overline{z})\big[(a \otimes \overline{x})(b \otimes \overline{y})\big] &= (\partial \otimes \overline{z})(ab \otimes \overline{x}\,\overline{y}) = \partial(ab) \otimes \overline{z}\,\overline{x}\,\overline{y} \\ &= \big[\partial(a)b + \varepsilon(z,x)a\partial(b)\big] \otimes \overline{z}\,\overline{x}\,\overline{y} \\ &= \partial(a)b \otimes \overline{z}\,\overline{x}\,\overline{y} + \varepsilon(z,x)a\partial(b) \otimes \overline{z}\,\overline{x}\,\overline{y} \\ &= \partial(a)b \otimes \overline{z}\,\overline{x}\,\overline{y} + \varepsilon'(z,x)a\partial(b) \otimes \overline{x}\,\overline{z}\,\overline{y} \\ &= \big[(\partial \otimes \overline{z})(a \otimes \overline{x})\big](b \otimes \overline{y}) + \varepsilon'(z,x)(a \otimes \overline{x})\big[(\partial \otimes \overline{z})(b \otimes \overline{y})\big],\end{aligned}$$

as required. More to the point, we have

Lemma 4.4. *Let A be a Γ-graded associative K-algebra. Then the map $T\colon \Delta \mapsto \Delta \boxtimes K^\tau[\Gamma]$ yields a one-to-one correspondence from the family of graded vector spaces of ε-commutative, ε-derivations of A to the family of graded vector spaces of ε'-commutative, ε'-derivations of $T(A) = A \boxtimes K^\tau[\Gamma]$. Here T^{-1} is essentially given by $T^{-1}\colon \Delta' \mapsto \Delta' \boxtimes K^{\tau^{-1}}[\Gamma]$.*

Proof. By the above, we know that $\Delta' = \Delta \boxtimes K^\tau[\Gamma]$ acts as ε'-derivations on the algebra $A' = A \boxtimes K^\tau[\Gamma]$, and this action is clearly faithful. Furthermore, if Δ is ε-commutative, then Lemma 4.2(i) implies that Δ' is ε'-commutative. Finally, if we let $S(\Delta') = \Delta' \boxtimes K^{\tau^{-1}}[\Gamma]$ and $S(A') = A' \boxtimes K^{\tau^{-1}}[\Gamma]$, then

$$\begin{aligned}S(T(\Delta)) &= (\Delta \boxtimes K^\tau[\Gamma]) \boxtimes K^{\tau^{-1}}[\Gamma] \cong \Delta \boxtimes K[\Gamma], \\ S(T(A)) &= (A \boxtimes K^\tau[\Gamma]) \boxtimes K^{\tau^{-1}}[\Gamma] \cong A \boxtimes K[\Gamma] \cong A,\end{aligned}$$

and $\Delta \boxtimes K[\Gamma]$ acts on $A \boxtimes K[\Gamma]$ in precisely the same way that Δ acts on A. Thus, S is essentially equal to T^{-1}. $\square$

Now it is clear from the preceding two lemmas that the maps $\_ \boxtimes K^\tau[\Gamma]$ yield a one-to-one correspondence from the isomorphism classes of ε-Lie color algebras of Witt type to the isomorphism classes of ε'-Lie color algebras of Witt type. To make this precise, we first need the following variant of [S, Theorem 2].

Lemma 4.5. *The map* $T\colon \mathfrak{g} \mapsto \mathfrak{g} \boxtimes K^\tau[\Gamma]$ *yields a one-to-one correspondence from the isomorphism classes of* Γ*-graded* ε*-Lie color algebras to the isomorphism classes of* Γ*-graded* ε'*-Lie color algebras. Here the color brackets are related by*

$$[a \otimes \overline{x}, b \otimes \overline{y}]' = [a,b] \otimes \overline{x}\,\overline{y} \qquad \text{for all } x,y \in \Gamma,\ a \in \mathfrak{g}_x,\ b \in \mathfrak{g}_y,$$

and T^{-1} *is given by* $T^{-1}\colon \mathfrak{g}' \mapsto \mathfrak{g}' \boxtimes K^{\tau^{-1}}[\Gamma]$*. Furthermore,* $\mathfrak{g}$ *is simple if and only if* $T(\mathfrak{g})$ *is simple.*

Proof. Let $\mathfrak{g}$ be an ε-Lie color algebra and let $U(\mathfrak{g})$ denote its universal enveloping algebra. Then $\mathfrak{g} \subseteq U(\mathfrak{g})$ and in fact, $\mathfrak{g}$ is an ε-Lie color subalgebra of $\mathcal{L}_\varepsilon(U(\mathfrak{g}))$, the ε-Lie color algebra of $U(\mathfrak{g})$. Now observe that $\mathfrak{g} \boxtimes K^\tau[\Gamma]$ is contained in the associative algebra $U(\mathfrak{g}) \boxtimes K^\tau[\Gamma]$ and we show that it is an ε'-Lie color subalgebra of $\mathcal{L}_{\varepsilon'}(U(\mathfrak{g}) \boxtimes K^\tau[\Gamma])$. Indeed, if $x, y \in \Gamma$, $a \in \mathfrak{g}_x$, $b \in \mathfrak{g}_y$, and if $[\ ,\ ]'$ denotes the ε'-Lie bracket, then Lemma 4.2(i) yields

$$\begin{aligned}
[a \otimes \overline{x}, b \otimes \overline{y}]' &= (a \otimes \overline{x})(b \otimes \overline{y}) - \varepsilon'(a,b)(b \otimes \overline{y})(a \otimes \overline{x}) \\
&= (ab \otimes \overline{x}\,\overline{y}) - \varepsilon(a,b)\mu_\tau(x,y)(ba \otimes \overline{y}\,\overline{x}) \\
&= (ab \otimes \overline{x}\,\overline{y}) - \varepsilon(a,b)(ba \otimes \overline{x}\,\overline{y}) \\
&= \big(ab - \varepsilon(a,b)ba\big) \otimes \overline{x}\,\overline{y} = [a,b] \otimes \overline{x}\,\overline{y}.
\end{aligned}$$

In otherwords, we have shown that $\mathfrak{g}' = \mathfrak{g} \boxtimes K^\tau[\Gamma]$ is an ε'-Lie color algebra with ε'-Lie bracket as above. Furthermore, if $\mathfrak{H}$ is a color ideal of $\mathfrak{g}$, then since $\mathfrak{H}$ must be Γ-graded, it is clear that $\mathfrak{H}' = \mathfrak{H} \boxtimes K^\tau[\Gamma]$ is a color ideal of $\mathfrak{g}'$. Finally, since $K^\tau[\Gamma] \boxtimes K^{\tau^{-1}}[\Gamma] \cong K[\Gamma]$, it is easy to see that $\mathfrak{g}' \boxtimes K^{\tau^{-1}}[\Gamma]$, with the appropriate Lie bracket, is naturally isomorphic to $\mathfrak{g}$. □

Putting this all together, we quickly obtain

Proposition 4.6. *The map* $T\colon \mathfrak{g} \mapsto \mathfrak{g} \boxtimes K^\tau[\Gamma]$ *yields a one-to-one correspondence from the isomorphism classes of* ε*-Lie color algebras of Witt type to the isomorphism classes of* ε'*-Lie color algebras of Witt type. Specifically, if* $\mathfrak{g} = A\Delta$*, then* $\mathfrak{g}' = \mathfrak{g} \boxtimes K^\tau[\Gamma] \cong A'\Delta'$ *where* $A' = A \boxtimes K^\tau[\Gamma]$ *and* $\Delta' = \Delta \boxtimes K^\tau[\Gamma]$*. Furthermore,* $\mathfrak{g}$ *is simple if and only if* $\mathfrak{g}'$ *is simple.*

Proof. Let A be an ε-commutative, graded associative K-algebra and let Δ be an ε-commutative, graded K-vector space of ε-derivations of A. If $\mathfrak{g} = A\Delta$, then it suffices to show that $\mathfrak{g}' = \mathfrak{g} \boxtimes K^\tau[\Gamma]$ is naturally isomorphic to $A'\Delta'$ where A' and Δ' are given by Lemmas 4.3 and 4.4. For this, it is best to view the construction of $A\Delta$ as formal multiplication.

Let $x, y, z, w \in \Gamma$, let $a \in A_x$, $\alpha \in \Delta_y$, $b \in A_z$ and $\beta \in \Delta_w$. Then $a' = a \otimes \overline{x}$ and $b' = b \otimes \overline{z}$ are elements of A', while $\alpha' = \alpha \otimes \overline{y}$ and $\beta' = \beta \otimes \overline{w}$ are in Δ'. For convenience, write $\overline{xy} = s\overline{x}\,\overline{y}$ and $\overline{zw} = t\overline{z}\,\overline{w}$ where $s = \tau(x,y)^{-1}$ and $t = \tau(z,w)^{-1}$. Then, we have

$$(a\alpha)' = a\alpha \otimes \overline{xy} = s(a\otimes \overline{x})(\alpha\otimes\overline{y}) = sa'\alpha'$$
$$(b\beta)' = b\beta \otimes \overline{zw} = t(b\otimes \overline{z})(\beta\otimes\overline{w}) = tb'\beta'.$$

In particular, if $[[\ ,\ ]]'$ denotes the ε'-Lie bracket of $A'\Delta'$, then

$$[[(a\alpha)', (b\beta)']]' = st[[a'\alpha', b'\beta']]' = st\big(a'\alpha'(b')\beta' - \varepsilon'(a'\alpha', b'\beta')b'\beta'(a')\alpha'\big).$$

Next, observe that

$$\begin{aligned}(st)a'\alpha'(b')\beta' &= (st)(a\otimes\overline{x})(\alpha(b)\otimes\overline{y}\,\overline{z})(\beta\otimes\overline{w})\\ &= a\alpha(b)\beta\otimes(s\overline{x}\,\overline{y})(t\overline{z}\,\overline{w}) = a\alpha(b)\beta\otimes\overline{xy}\,\overline{zw},\end{aligned}$$

and

$$\begin{aligned}&(st)\varepsilon'(a'\alpha', b'\beta')b'\beta'(a')\alpha'\\ &\quad= \varepsilon(a\alpha, b\beta)\mu_\tau(xy, zw)(st)(b\otimes\overline{z})(\beta(a)\otimes\overline{w}\,\overline{x})(\alpha\otimes\overline{y})\\ &\quad= \varepsilon(a\alpha, b\beta)\mu_\tau(xy, zw)\big(b\beta(a)\alpha\otimes(t\overline{z}\,\overline{w})(s\overline{x}\,\overline{y})\big)\\ &\quad= \varepsilon(a\alpha, b\beta)\big(b\beta(a)\alpha\otimes\mu_\tau(xy, zw)\overline{zw}\,\overline{xy}\big)\\ &\quad= \varepsilon(a\alpha, b\beta)(b\beta(a)\alpha\otimes\overline{xy}\,\overline{zw}).\end{aligned}$$

Thus, by the above computations and the previous lemma,

$$\begin{aligned}[[(a\alpha)', (b\beta)']]' &= \big(a\alpha(b)\beta - \varepsilon(a\alpha, b\beta)b\beta(a)\alpha\big)\otimes\overline{xy}\,\overline{zw}\\ &= [a\alpha, b\beta]\otimes\overline{xy}\,\overline{zw} = [(a\alpha)', (b\beta)']',\end{aligned}$$

so $\mathfrak{g}' \cong A'\Delta'$ as ε'-Lie color algebras. Lemma 4.5 now yields the result. □

Of course, the payoff here comes from the discoloration functor. If $\varepsilon\colon \Gamma\times\Gamma\to K^\bullet$ is a bicharacter, let Γ_+^ε denote the kernel of the parity function $x\mapsto\varepsilon(x,x)$. Then we have

Theorem 4.7. *Let $\varepsilon, \varepsilon'\colon \Gamma\times\Gamma\to K^\bullet$ be two bicharacters with $\Gamma_+^\varepsilon = \Gamma_+^{\varepsilon'}$. Then there is a one-to-one correspondence between the isomorphism classes of (simple) ε-Lie color algebras of Witt type and the isomorphism classes of (simple) ε'-Lie color algebras of Witt type.*

Proof. If $\Gamma_+^\varepsilon = \Gamma_+^{\varepsilon'}$, then it follows from [S, Lemma 2] and [Po, Theorem] that there exists a 2-cocycle $\tau\colon\Gamma\times\Gamma\to K^\bullet$ with $\varepsilon' = \varepsilon\mu_\tau$ (see Corollary 5.2 of this paper). Now apply Proposition 4.6. □

Of course, one can view the preceding result in two ways. In the positive direction, it offers a method of constructing simple Lie color algebras of Witt type from the corresponding super algebras. In the negative direction, it indicates that there will be no new surprises.

§5. Appendix

At present, the result of [Po] quoted in the proof of Theorem 4.7 seems to require that the field K be algebraically closed. Fortunately, we do not need as strong a conclusion as is contained in [Po, Theorem], where it is shown that a 2-cocycle exists which is multiplicative in each factor. Indeed, we just require the existence of a 2-cocycle, and this can be proved without any assumptions on the field. For the sake of completeness, we offer an elementary demonstration of this fact, changing notation somewhat and using a group theoretic argument.

Proposition 5.1. *Let A and Z be multiplicative abelian groups and let $\varepsilon\colon A\times A\to Z$ be a map. The following are equivalent.*

i. $\varepsilon(ab,c)=\varepsilon(a,c)\varepsilon(b,c)$, $\varepsilon(b,a)=\varepsilon(a,b)^{-1}$ *and* $\varepsilon(a,a)=1$ *for all* $a,b,c\in A$.
ii. *There exists a group G with $Z\subseteq \mathbb{Z}(G)$, $G/Z=A$ and such that the commutator map $(\ ,\ )\colon G/Z\times G/Z\to Z$ is equal to ε.*
iii. *There exists a 2-cocyle $\tau\colon A\times A\to Z$ with $\varepsilon(a,b)=\tau(a,b)/\tau(b,a)$ for all $a,b\in A$.*

Proof. We will show that (ii) is equivalent to (i) and to (iii). Most of this is routine. The only nontrivial argument occurs in the proof of implication (i) $\Rightarrow$ (ii).

Suppose first that (ii) holds. Then G is a class 2 group and consequently the commutator map $G\times G\to Z$ given by $x\times y\mapsto (x,y)=x^{-1}y^{-1}xy$ satisfies $(xy,z)=(x,z)(y,z)$, $(y,x)=(x,y)^{-1}$ and $(x,x)=1$ for all $x,y,z\in G$. Furthermore, (x,y) is constant on the cosets of Z. Thus the commutator gives rise to a map from $A\times A=G/Z\times G/Z$ to Z and if this map is ε, then ε satisfies the conditions of (i).

Next, for each $a\in A$, choose $\overline{a}\in G$ with $Z\overline{a}=a\in G/Z=A$. Then, for all $a,c\in A$, we have $\overline{a}\,\overline{c}=\tau(a,c)\overline{ac}$ with $\tau\colon A\times A\to Z$ a 2-cocycle. Furthermore, since $\overline{c}\,\overline{a}=\tau(c,a)\overline{ca}$ and $\overline{ca}=\overline{ac}$, we obtain

$$
\begin{aligned}
(\overline{a},\overline{c}) &= \overline{a}^{-1}\overline{c}^{-1}\overline{a}\,\overline{c}=(\overline{c}\,\overline{a})^{-1}\overline{a}\,\overline{c}\\
&=\tau(a,c)\tau(c,a)^{-1}\overline{ca}^{-1}\overline{ac}=\tau(a,c)\tau(c,a)^{-1}.
\end{aligned}
$$

Thus $\varepsilon(a,c)=(\overline{a},\overline{c})$ has the appropriate form and (iii) is proved.

Next, we show that (iii) $\Rightarrow$ (ii). To this end, let $\tau\colon A\times A\to Z$ be a 2-cocycle. Then τ defines a group G with central subgroup Z and with $G/Z=A$. Indeed, for each $a\in A$, there exists $\overline{a}\in G$ with $Z\overline{a}=a$ and with $\overline{a}\,\overline{c}=\tau(a,c)\overline{ac}$. As in the above displayed equation, we have $(\overline{a},\overline{c})=\tau(a,c)\tau(c,a)^{-1}=\varepsilon(a,c)$, by assumption, and thus the commutator map $(\ ,\ )\colon G/Z\times G/Z\to Z$ is equal to ε, as required.

It remains to prove that (i) $\Rightarrow$ (ii). To this end, we first embed Z in a divisible abelian group D. Next, we let X be the group generated by D and the symbols $\{\,x_a\mid a\in A\,\}$ with D central and with additional relations given by $(x_a,x_b)=\varepsilon(a,b)\in D$ for all $a,b\in A$. Since $\varepsilon(b,a)=\varepsilon(a,b)^{-1}$ and $\varepsilon(a,a)=1$, it is clear that X is just the central product of the free class 2 group on the symbols $\{\,x_a\mid a\in A\,\}$

with the group D, identifying (x_a, x_b) with $\varepsilon(a,b)$. In particular, X/D is isomorphic to the free abelian group on the elements Dx_a, and consequently there exists an epimorphism $\theta\colon X \to A$ given by $\theta(D) = 1$ and $\theta(x_a) = a$ for all $a \in A$.

Let $N = \ker\theta$, so that $X/N \cong A$. If $x = d\prod_i x_{a_i}^{e_i} \in N$ with $d \in D$, then $1 = \theta(x) = \prod_i a_i^{e_i}$. Hence, for all $b \in A$, we have

$$\begin{aligned}(x, x_b) &= (d\textstyle\prod_i x_{a_i}^{e_i}, x_b) = \prod_i (x_{a_i}, x_b)^{e_i}\\ &= \textstyle\prod_i \varepsilon(a_i, b)^{e_i} = \varepsilon(\prod_i a_i^{e_i}, b) = \varepsilon(1, b) = 1,\end{aligned}$$

since ε is multiplicative in its first variable. It follows that N is central in X and, in particular, that N is abelian. But $N \supseteq D$ and D is divisible, so $N = D \times M$ for some subgroup M. Furthermore, since M is central in X, it is normal and we can set $H = X/M$. Note that $D = N/M \triangleleft X/M = H$ and $H/D \cong X/N = A$. Indeed, since $(x_a, x_b) = \varepsilon(a,b) \in D$, it follows that the commutator map $(\ ,\ )\colon H/D \times H/D \to D$ is equal to ε.

Finally, note that $H' = (H,H) \subseteq Z$ and that Z is central in H. Thus H/Z is an abelian group containing the divisible group D/Z as a subgroup. It therefore follows that $H/Z = G/Z \times D/Z$ for some subgroup G of H. Of course, $G/Z \cong H/D \cong A$ and it is clear that $(\ ,\ )\colon G/Z \times G/Z \to Z$ is equal to ε. Thus (ii) is proved. □

Reverting to the notation of the previous section, we have

Corollary 5.2. *Let $\varepsilon, \varepsilon'\colon \Gamma \times \Gamma \to K^\bullet$ be two bicharacters with $\Gamma_+^\varepsilon = \Gamma_+^{\varepsilon'}$. Then there exists a 2-cocycle $\tau\colon \Gamma \times \Gamma \to K^\bullet$ with*

$$\varepsilon'(x,y) = \varepsilon(x,y)\mu_\tau(x,y) = \varepsilon(x,y)\tau(x,y)/\tau(y,x) \qquad \textit{for all } x, y \in \Gamma.$$

Proof. It is clear that $\varepsilon'\varepsilon^{-1}\colon \Gamma \times \Gamma \to K^\bullet$ is a bicharacter with $\varepsilon'\varepsilon^{-1}(x,x) = 1$ for all $x \in \Gamma$. Thus, by (i) $\Rightarrow$ (iii) of the previous proposition, with $A = \Gamma$ and $Z = K^\bullet$, there exists a 2-cocycle $\tau\colon \Gamma \times \Gamma \to K^\bullet$ with $\varepsilon'(x,y)\varepsilon(x,y)^{-1} = \tau(x,y)\tau(y,x)^{-1} = \mu_\tau(x,y)$ for all $x, y \in \Gamma$, and the result follows. □

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