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MULTIPLICATIVE JORDAN DECOMPOSITION IN GROUP RINGS OF 3-GROUPS

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In this paper, we essentially classify those finite 3-groups G having integral group rings with the multiplicative Jordan decomposition property. If G is abelian, then it is clear that $\mathbb{Z}[G]$ satisfies MJD. Thus, we are only concerned with the nonabelian case. Here we show that $\mathbb{Z}[G]$ has the MJD property for the two nonabelian groups of order 3^3. Furthermore, we show that there are at most three other specific nonabelian groups, all of order 3^4, with $\mathbb{Z}[G]$ having the MJD property. Unfortunately, we are unable to decide which, if any, of these three satisfies the appropriate condition.

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1. Introduction

Let $\mathbb{Q}[G]$ denote the rational group algebra of the finite group G. Since $\mathbb{Q}$ is a perfect field, every element a of $\mathbb{Q}[G]$ has a unique additive Jordan decomposition $a = a_s + a_n$, where a_s is a semisimple element and where a_n commutes with a_s and is nilpotent. If a is a unit, then a_s is also invertible and $a = a_s(1 + a_s^{-1}a_n)$ is a product of a semisimple unit a_s and a commuting unipotent unit $a_u = 1 + a_s^{-1}a_n$.

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This is the unique multiplicative Jordan decomposition of a. Following [AHP] and [HPW], we say that $\mathbb{Z}[G]$ has the multiplicative Jordan decomposition property (MJD) if for every unit a of $\mathbb{Z}[G]$, its semisimple and unipotent parts are both contained in $\mathbb{Z}[G]$. For simplicity, we say that G satisfies MJD if its integral group ring $\mathbb{Z}[G]$ has that property.

If G is abelian or a Hamiltonian 2-group, then every element of $\mathbb{Q}[G]$ is semisimple. Thus every unit a of $\mathbb{Z}[G]$ is equal to its semisimple part and consequently $\mathbb{Z}[G]$ trivially satisfies MJD. In the non-Dedekind case, it appears that the MJD property is relatively rare. Indeed, the papers [AHP] and [HPW] have shown that $\mathbb{Z}[G]$ and $\mathbb{Q}[G]$ must be quite restrictive. For example, we have the following, with part (i) from [AHP, Theorem 4.1] and part (ii) from [HPW, Corollary 9].

Theorem 1.1. *Let G have the multiplicative Jordan decomposition property.*

- *i. If the matrix ring $\mathrm{M}_n(D)$ over the division ring D is a Wedderburn component of $\mathbb{Q}[G]$, then $n \leq 3$.*
- *ii. If z is a nilpotent element of $\mathbb{Z}[G]$ and e is a central idempotent of $\mathbb{Q}[G]$, then $ze \in \mathbb{Z}[G]$.*

Using this and numerous clever arguments, paper [HPW] was able to determine all nonabelian 2-groups that satisfy MJD. Specifically, these are the two nonabelian groups of order 8, five groups of order 16, four groups of order 32, and only the Hamiltonian groups of larger order. In this paper, we build on the work of [HPW], using variants of many of the same arguments, to determine all nonabelian 3-groups satisfying MJD. These are the two nonabelian groups of order 3^3 and at most three groups of order 3^4. Unfortunately, we are not able to decide which, if any, of the latter three groups have the appropriate property.

As will be apparent, eliminating groups is more difficult in the 2-group case than for 3-groups because of the presence of nonabelian Dedekind groups and of quaternion division algebras. On the other hand, proving that certain group rings have the MJD property is easier in the 2-group case because the Wedderburn components $\mathrm{M}_n(D)$ of $\mathbb{Q}[G]$ have $n \leq 2$. We remark that our reformulation of some of the arguments of [HPW], in particular our Proposition 2.5 and Lemma 2.6, can be used to eliminate much of the special case analysis in [HPW].

Of course, a multiplicative Jordan decomposition might exist in an algebra over any field K, and we close this section with the following obvious result.

Lemma 1.2. *Let $R = R_1 \oplus R_2$ be the algebra direct sum of the two finite dimensional K-algebras R_1 and R_2, and let $a = (a_1, a_2) \in R_1 \oplus R_2$.*

- *i. a is a unit of R if and only if a_1 is a unit of R_1 and a_2 is a unit of R_2. Indeed, when this occurs then $a^{-1} = (a_1^{-1}, a_2^{-1})$.*
- *ii. $a = su$ is the multiplicative Jordan decomposition of the unit $a \in R$ with $s = (s_1, s_2)$ semisimple and with $u = (u_1, u_2)$ unipotent, if and only if $a_1 = s_1u_1$*

and $a_2 = s_2u_2$ are the corresponding multiplicative Jordan decompositions of a_1 and a_2.

iii. *If a_1 is a semisimple unit of R_1 and a_2 is a unipotent unit of R_2, then (a_1, a_2) is a unit of R with multiplicative Jordan decomposition given by $su = (a_1, 1)(1, a_2)$.*

2. Groups of order $\geq 3^4$

The main result of this section restricts the possible 3-groups whose integral group rings satisfy MJD. Indeed, we prove

Theorem 2.1. *Let G be a finite nonabelian 3-group of order $\geq 3^4$. If $\mathbb{Z}[G]$ has the MJD property, then G can only be one of three specific groups of order 3^4, namely*

i. *the central product of a cyclic group of order 9 with the nonabelian group of order 27 and period 3, or*
ii. *the group generated by x, y and z subject to the relations $x^9 = y^3 = 1$, $xy = yx$, $x^z = xy$, $y^z = yx^{-3}$ and $z^3 = x^3$, or*
iii. *the semidirect product $G = X \rtimes Y$, where X and Y are cyclic of order 9.*

We start the proof by constructing a particularly useful unit in $\mathbb{Z}[\varepsilon]$, where ε is a primitive complex 9th root of unity.

Lemma 2.2. *Let ε be a complex primitive 9th root of unity and set $\alpha = \varepsilon + \varepsilon^{-1}$. Then $1 - 3\alpha$ is a unit in $\mathbb{Z}[\varepsilon]$ with inverse $1 + 3\alpha + 9\alpha^2 - 27$.*

Proof. Since ε^3 is a primitive cube root of unity, we have $\varepsilon^3 + \varepsilon^{-3} = -1$. Thus

$$\alpha^3 = \varepsilon^3 + \varepsilon^{-3} + 3(\varepsilon + \varepsilon^{-1}) = -1 + 3\alpha$$

and hence $1 + \alpha^3 = 3\alpha$. It now follows that

$$(1 - 3\alpha)(1 + 3\alpha + 9\alpha^2 - 27) = 1 - 27(1 + \alpha^3 - 3\alpha) = 1$$

and the proof is complete. □

Since we will have to raise this unit to a suitable power of 3, we need the following reasonably well known result.

Lemma 2.3. *Let p be an odd prime and let α and β be elements of the commutative ring R. Then for all integers $k \geq 0$ we have*

$$(1 + p\alpha + p^2\beta)^{p^k} = 1 + p^{k+1}\alpha + p^{k+2}\beta_k$$

for some $\beta_k \in R$.

Proof. We proceed by induction on k. The $k=0$ result is given with $\beta_0=\beta$. Now suppose the result holds for k. Then

$$\begin{aligned}(1+p\alpha+p^2\beta)^{p^{k+1}} &= (1+p^{k+1}\alpha+p^{k+2}\beta_k)^p = [1+p^{k+1}(\alpha+p\beta_k)]^p\\ &= \sum_{i=0}^{p}\binom{p}{i}p^{i(k+1)}(\alpha+p\beta_k)^i.\end{aligned}$$

But $p^{i(k+1)}\equiv 0 \bmod p^{k+3}R$ for $i\geq 3$ and $\binom{p}{2}p^{2(k+1)}\equiv 0 \bmod p^{k+3}R$ since p divides the binomial coefficient. Thus

$$(1+p\alpha+p^2\beta)^{p^{k+1}}\equiv 1+pp^{k+1}(\alpha+p\beta_k)\equiv 1+p^{k+2}\alpha \mod p^{k+3}R,$$

as required. □

As usual, if X is a subset of G, we write $\widehat{X}$ for the sum of the members of X in $\mathbb{Z}[G]$. Furthermore, if H is a subgroup of G, we will write $e_H=\widehat{H}/|H|$ for the principal idempotent in $\mathbb{Q}[H]$ determined by H. If $H\triangleleft G$, then e_H is central in $\mathbb{Q}[G]$ and, as is well known, $e_H\mathbb{Q}[G]$ is naturally isomorphic to $\mathbb{Q}[G/H]$. We now obtain the $p=3$ analog of [HPW, Proposition 22].

Proposition 2.4. *Let G be a 3-group with $G'=Z$ central of order 3, and suppose that A is a normal abelian subgroup of G with G/A cyclic of order 9. If A has a subgroup C that is not normal in G, then $\mathbb{Z}[G]$ does not have MJD.*

Proof. Write $G=\langle A,g\rangle$ so that the image of $g\in G$ generates the cyclic group G/A of order 9. Furthermore, let $B=\langle A,g^3\rangle$ be the unique maximal subgroup of G properly larger than A. Since $G'=Z$ is central of order 3, it follows that g^3 is central in G (see, for example, Lemma 2.8) and hence B is also a normal abelian subgroup of G. In $\mathbb{Q}[G]$ define

$$e=e_A-e_B=(2-g^3-g^{-3})e_A/3.$$

Claim 1. *e is a central idempotent in $\mathbb{Q}[G]$ with $e\mathbb{Q}[G]$ equal to the cyclotomic field $\mathbb{Q}[\varepsilon]$, where $\varepsilon=eg$ is a primitive 9th root of unity.*

Proof. Since $e_A\mathbb{Q}[G]$ is naturally isomorphic to $\mathbb{Q}[G/A]$, it suffices for this claim to temporarily assume that $A=1$. Then $G=\langle g\rangle$ is cyclic of order 9 and $\mathbb{Q}[G]$ is isomorphic to the polynomial ring $\mathbb{Q}[\zeta]$ modulo the principal ideal (ζ^9-1). Since $\zeta^9-1=\Phi_1(\zeta)\Phi_3(\zeta)\Phi_9(\zeta)$ is the product of three irreducible cyclotomic polynomials, we have the algebra direct sum $\mathbb{Q}[G]=e_1\mathbb{Q}[G]+e_2\mathbb{Q}[G]+e_3\mathbb{Q}[G]$ where $e_1\mathbb{Q}[G]\cong\mathbb{Q}$, $e_2\mathbb{Q}[G]\cong\mathbb{Q}[\omega]$ with ω a primitive cube root of 1, and $e_3\mathbb{Q}[G]\cong\mathbb{Q}[\varepsilon]$. Furthermore, $e_G=e_1$ and $e_B=e_1+e_2$, so $e_1=e_G$, $e_2=e_B-e_G$, and $e_3=1-e_B$. Finally, note that

$$e_3=1-e_B=1-(1+g^3+g^{-3})/3=(2-g^3-g^{-3})/3$$

has the appropriate form. □

By assumption, G/A is abelian so $A \supseteq G' = Z$ and hence $|A| \geq 3$. For convenience, write $|A| = 3^{k+1}$ for some integer $k \geq 0$. Now define $\alpha = g + g^{-1} \in \mathbb{Z}[G]$ and set

$$u_1 = e(1 - 3\alpha)^{3^k} \in e\mathbb{Q}[G].$$

Claim 2. *u_1 is a semisimple unit in $e\mathbb{Q}[G]$ with inverse*

$$v_1 = e(1 + 3\alpha + 9\alpha^2 - 27)^{3^k}.$$

Proof. This is immediate from Lemma 2.2 and the preceding claim. □

Next, we study the complementary algebra direct summand $(1-e)\mathbb{Q}[G]$. To start with, since C is not normal in G, we know that C does not contain $G' = Z$. Thus, since $|Z| = 3$, we have $C \cap Z = 1$ and $CZ \cong C \times Z$. Let T be a set of coset representatives for CZ in A. Then CT is a full set of coset representatives for Z in A, and we define

$$\gamma = (2 - g^3 - g^{-3})\widehat{C}\widehat{T}\alpha \in \mathbb{Z}[G]$$

and

$$u_2 = (1-e)(1-\gamma) \in (1-e)\mathbb{Q}[G].$$

Claim 3. *$(1-e)\gamma$ has square 0. In particular, u_2 is a unipotent unit in $(1-e)\mathbb{Q}[G]$ with inverse $v_2 = (1-e)(1+\gamma)$.*

Proof. For convenience, set $\beta = (2 - g^3 - g^{-3})\widehat{C}\widehat{T}$ and let $h = g$ or g^{-1}. Since C is normal in the abelian group B, but not normal in G, it follows that h does not normalize C. On the other hand, C has index 3 in $ZC \triangleleft G$. Thus $CC^h = ZC$ and $\widehat{C}\widehat{C}^h = m\widehat{Z}\widehat{C}$ where $m = |C|/3$. In particular, $(\widehat{C}\widehat{T})(\widehat{C}\widehat{T})^h$ is divisible by $\widehat{Z}\widehat{C}\widehat{T} = \widehat{A}$, and hence $\beta\beta^h$ is divisible by $\widehat{Z}\widehat{C}\widehat{T}(2-g^3-g^{-3}) = \widehat{A}(2-g^3-g^{-3})$, a scalar multiple of the idempotent e.

It follows that $(1-e)\beta\beta^h = 0$ and hence $(1-e)\beta h^{-1}\beta = 0$. In particular, since $\alpha = g + g^{-1}$, we conclude that $(1-e)\beta\alpha\beta = 0$ and consequently $(1-e)\gamma = (1-e)\beta\alpha$ has square 0. The remaining comments concerning u_2 and v_2 are now clear. □

We know, of course, that $\mathbb{Q}[G]$ is naturally isomorphic to the algebra direct sum $e\mathbb{Q}[G] \oplus (1-e)\mathbb{Q}[G]$. However, to avoid confusion and direct sum notation, we will work entirely within $\mathbb{Q}[G]$.

Claim 4. *$u = u_1 + u_2$ is a unit in $\mathbb{Z}[G]$ with inverse $v = v_1 + v_2$. The semisimple part of u is $s = u_1 + (1-e)$ and its unipotent part is $t = e + u_2$. In particular, since neither s nor t is contained in $\mathbb{Z}[G]$, we conclude that G does not satisfy MJD.*

Proof. Let us use $\sigma \equiv \tau$ to indicate that the two elements of $\mathbb{Q}[G]$ differ by an element of $\mathbb{Z}[G]$. First, observe by Lemma 2.3 that

$$(1-3\alpha)^{3^k} = 1 - 3^{k+1}\alpha + 3^{k+2}\beta$$

for some $\beta \in \mathbb{Z}[G]$. Thus, since the denominator of e is $3|A| = 3^{k+2}$, it follows that $3^{k+2}\beta e \in \mathbb{Z}[G]$ and hence $u_1 \equiv (1 - 3^{k+1}\alpha)e$. Similarly, $v_1 \equiv (1 + 3^{k+1}\alpha)e$.

Next, since $\gamma \in \mathbb{Z}[G]$, we see that $u_2 = (1-e)(1-\gamma) \equiv -e(1-\gamma)$. Furthermore, since $\widehat{A}\widehat{C}\widehat{T} = \widehat{A}|A|/3 = \widehat{A}3^k$ and $\widehat{A}(2-g^3-g^{-3})^2 = 3\widehat{A}(2-g^3-g^{-3})$, we see that $e\gamma = 3^{k+1}e\alpha$ and $u_2 \equiv -e(1-3^{k+1}\alpha)$. Similarly, $v_2 \equiv -e(1+3^{k+1}\alpha)$. Thus

$$u = u_1 + u_2 \equiv e(1 - 3^{k+1}\alpha) - e(1 - 3^{k+1}\alpha) \equiv 0$$

and

$$v = v_1 + v_2 \equiv e(1 + 3^{k+1}\alpha) - e(1 + 3^{k+1}\alpha) \equiv 0.$$

In other words, we have shown that $u, v \in \mathbb{Z}[G]$ and since $uv = 1$ by Lemma 1.2, we conclude that u is a unit in $\mathbb{Z}[G]$. Furthermore, by Lemma 1.2 again, $s = u_1 + (1-e)$ is the semisimple part of u and $t = e + u_2$ is the unipotent part. But

$$\begin{aligned} s = u_1 + (1-e) &\equiv (1 - 3^{k+1}\alpha)e + (1-e) \\ &= 1 - 3^{k+1}\alpha e \equiv -3^{k+1}\alpha e = (g+g^{-1})(g^3+g^{-3}-2)\widehat{A}/3 \end{aligned}$$

and the latter element is clearly not in $\mathbb{Z}[G]$ since all the coefficients of the group elements in the coset g^4A are equal to $1/3$.

Similarly,

$$t = e + u_2 \equiv e - e(1-\gamma) = e\gamma = 3^{k+1}e\alpha \not\equiv 0.$$

Thus u is a unit in $\mathbb{Z}[G]$ whose multiplicative Jordan factors are not in $\mathbb{Z}[G]$ and we conclude that G does not satisfy MJD. □

In view of the preceding claim, the result follows. □

The following extends [HPW, Corollary 10]. This formulation could be used to simplify much of the special case analyses of that paper.

Proposition 2.5. *Let G have MJD and let $N \triangleleft G$. If Y is any subgroup of G, then either $Y \supseteq N$ or $YN \triangleleft G$.*

Proof. Suppose $H = YN$ is not normal in G and let g be an element of G not in the normalizer of H. Since $N^g = N \subseteq H$, we must have $Y^g \not\subseteq H$. In particular, there exists $y \in Y$ with $y^g \notin H$. Set $\alpha = (1-y)g\widehat{Y} \in \mathbb{Z}[G]$ and note that α is nilpotent since $\widehat{Y}(1-y) = 0$. Furthermore, $e = \widehat{N}/|N|$ is a central idempotent of $\mathbb{Q}[G]$. Thus, Theorem 1.1(ii) implies that $\alpha e \in \mathbb{Z}[G]$. Now $\widehat{Y}\widehat{N} = \widehat{H}\cdot|Y|\cdot|N|/|H| = \widehat{H}\cdot|Y \cap N|$, and hence $\alpha e = (1-y)g\widehat{H}\cdot|Y \cap N|/|N|$. Note that the support of αe consists of two cosets of H, namely gH and ygH, and these cosets are distinct since otherwise $H = g^{-1}ygH$ and $y^g \in H$. It follows that the coefficient of g in αe is equal to $0 < |Y \cap N|/|N| \le 1$. But $\alpha e \in \mathbb{Z}[G]$, so $|Y \cap N|/|N| = 1$ and $Y \supseteq N$. □

This has numerous consequences, most notably

Lemma 2.6. *Let G have MJD and let N be a noncyclic normal subgroup of G. Then G/N is a Dedekind group. In particular, if G/N has odd order, then this factor group is abelian.*

Proof. If Y is a cyclic subgroup of G, then Y cannot contain N. Thus, Proposition 2.5 implies that $YN \triangleleft G$, and hence $YN/N \triangleleft G/N$. It follows that all cyclic subgroups of G/N are normal, and hence all subgroups of G/N are normal. By definition, G/N is a Dedekind group. In particular, if G/N has odd order, then this factor group must be abelian. □

In view of the above, the study of MJD groups of odd order should be simpler than the 2-group case. To start with, we have

Lemma 2.7. *If G is a nonabelian 3-group with MJD, then $\mathfrak{Z}(G)$, the center of G, has rank at most 2.*

Proof. If $\mathfrak{Z}(G)$ has rank ≥ 3, then $\mathfrak{Z}(G)$ contains an elementary abelian subgroup of order 27. It then follows that G has three central subgroups Z_1, Z_2 and Z_3, each elementary abelian of order 9, and with $Z_1 \cap Z_2 \cap Z_3 = 1$. By the previous result, G/Z_i is abelian, so the commutator subgroup G' is contained in each Z_i. Hence $G' \subseteq Z_1 \cap Z_2 \cap Z_3 = 1$, a contradiction. □

We consider the two cases, where the rank of $\mathfrak{Z}(G)$ is 1 or 2, separately. But first, we mention an elementary group-theoretic result. It is an immediate consequence of the theory of regular p-groups, but it is easy enough to prove directly.

Lemma 2.8. *Let G be a p-group with commutator subgroup G' central of period p. Then $G/\mathfrak{Z}(G)$ has period p. Furthermore, if $p > 2$, then the pth power map $x \mapsto x^p$ is a homomorphism from G to $\mathfrak{Z}(G)$.*

Proof. Let $x, y \in G$. Then $y^x = yz$ for some $z \in G'$. In particular, z is central and $z^p = 1$. Hence $(y^p)^x = (yz)^p = y^p z^p = y^p$, and y^p is central. Furthermore, if $p > 2$, then we have

$$y^{x^{p-1}} y^{x^{p-2}} \cdots y^x y = yz^{p-1} yz^{p-2} \cdots yzy = y^p z^{p(p-1)/2} = y^p$$

since $(p-1)/2$ is an integer. Thus

$$(xy)^p = (xy)(xy)\cdots(xy)(xy) = x^p \cdot y^{x^{p-1}} y^{x^{p-2}} \cdots y^x y = x^p y^p,$$

as required. □

We now consider the rank 1 case.

Lemma 2.9. *Let G be a nonabelian 3-group with MJD and suppose that $\mathfrak{Z}(G)$ is cyclic. If $|G| > 27$, then $|G| = 81$ and G is either* *i. the central product of a cyclic group of order* 9 *with the nonabelian group of order* 27 *and period* 3*, or*

ii. the group generated by x, y and z subject to the relations $x^9 = y^3 = 1$, $xy = yx$, $x^z = xy$, $y^z = yx^{-3}$ and $z^3 = x^3$.

In particular, G is a group of type (*i*) *or* (*ii*) *in Theorem* 2.1*.*

Proof. Let Z be the unique subgroup of $\mathfrak{Z}(G)$ of order 3. Since G is not cyclic, we know from [R, Lemma 3] that G has a normal abelian subgroup B of type $(3,3)$. Thus $B = Z \times J$, where J is a noncentral, and hence nonnormal, subgroup of G of order 3. Since $|\mathrm{Aut}(B)|$ is divisible by 3, but not 9, it follows that $C = \mathfrak{C}_G(B)$, the centralizer of B in G, is a normal subgroup of G of index 3. Of course, $C \supseteq B$.

Claim 1. *G/B is abelian of period* 3*.*

Proof. Lemma 2.6 implies that G/B is abelian. Suppose, by way of contradiction, that G/B has period ≥ 9. Since this abelian group is generated by its elements not in C/B, it follows that some element in $(G/B) \setminus (C/B)$ has order ≥ 9. In other words, there exists an element $g \in G \setminus C$ with $g^3 \notin B$.

Now let H be the subgroup of G generated by B and g, so that $H = B\langle g\rangle$. Since B/Z is central in G/Z, it follows that H/Z is abelian, so $H' \subseteq Z$. On the other hand, g does not centralize B, so H is nonabelian, and hence $H' = Z$. Next, since G/C has order 3, it follows that g^3 centralizes B. In particular, $A = B\langle g^9\rangle$ is a normal abelian subgroup of H with H/A cyclic of order 9. Note that g does not centralize J, so it does not normalize J. Thus A contains J, a nonnormal subgroup of H. Proposition 2.4 now implies that H does not have MJD. But MJD is clearly inherited by subgroups, so this is the required contradiction. □

Claim 2. *$C = \mathfrak{C}_G(B)$ is abelian.*

Proof. To start with, Theorem 1.1(i) asserts that all Wedderburn components of $\mathbb{Q}[G]$ have degree ≤ 3. Furthermore, by Roquette's theorem [R, Satz 1], each of the division rings occurring in these Wedderburn components is a field. In other words, every irreducible representation θ of $\mathbb{Q}[G]$ is an epimorphism $\theta\colon \mathbb{Q}[G] \to \mathrm{M}_k(F)$, where F is a field and $k \leq 3$. Since $\mathbb{Q}[G]$ is semisimple, there exists such a representation θ with Z not in the kernel of the corresponding group homomorphism $\theta\colon G \to \mathrm{GL}_k(F)$. Indeed, since every nontrivial normal subgroup of G meets $\mathfrak{Z}(G)$ nontrivially and hence contains Z, we conclude that θ is faithful on G.

Let $V = F^k$ be the k-dimensional F-vector space acted upon by $\mathrm{GL}_k(F)$ and hence by G. Then irreducibility implies that no proper subspace of V is G-stable. In particular, the Z-fixed point space $\mathfrak{C}_V(Z) = \{v \in V \mid Zv = v\}$ satisfies $\mathfrak{C}_V(Z) = 0$. On the other hand, B is abelian of type $(3,3)$, so B cannot act in a fixed-point-free manner on V. Thus B has a subgroup L of order 3 with $\mathfrak{C}_V(L) \neq 0$. Of course,

$\mathfrak{C}_V(L)$ is an F-subspace of V and $L \neq Z$. Thus L has $|G : C| = 3$ distinct G-conjugates, say L_1, L_2, L_3, all contained in B. Now, for all $i = 1, 2, 3$, the fixed point space $\mathfrak{C}_V(L_i)$ is G-conjugate to $\mathfrak{C}_V(L)$, and therefore all such subspaces have the same dimension.

Next, for $i \neq j$, we have $L_i L_j = B \supseteq Z$, so $\mathfrak{C}_V(L_i) \cap \mathfrak{C}_V(L_j) \subseteq \mathfrak{C}_V(Z) = 0$. Thus since $\dim_F V = k \leq 3$, it follows that $\mathfrak{C}_V(L_i)$ has dimension 1 for all i. Furthermore, $W = \sum_{i=1}^{3} \mathfrak{C}_V(L_i)$ is a nonzero G-stable subspace of V, so $W = V$. Finally, note that C centralizes L_i, so it acts on $\mathfrak{C}_V(L_i)$. Thus θ induces a homomorphism θ_i from C to the general linear group on the 1-dimensional space $\mathfrak{C}_V(L_i)$. In other words, $\theta_i(C) \subseteq \mathrm{GL}_1(F)$ and hence $C/\ker\theta_i \cong \theta_i(C)$ is abelian. But $W = V$ implies that $\bigcap_{i=1}^{3} \ker\theta_i = 1$, so C is abelian, and the claim is proved. □

Claim 3. *G/Z has period 3, C is abelian of type $(9, 3)$ and $|G| = 81$.*

Proof. We already know that C is abelian and that G/B has period 3. Fix $x \in G \backslash C$ and note that $G = \langle C, x \rangle$, so $\mathfrak{C}_C(x) = \mathfrak{Z}(G)$. Furthermore, $Z = B \cap \mathfrak{Z}(G)$.

We show that G/Z has period 3. Suppose first that $g \in G \setminus C$. The $G = \langle C, g \rangle$ and $g^3 \in B$ commutes with both C and g. Thus $g^3 \in B \cap \mathfrak{Z}(G) = Z$. On the other hand, if $g \in C$, then $g^x = gb$ for some $b \in B$. Thus since C is abelian and B has period 3, we see that $(g^3)^x = (g^x)^3 = g^3 b^3 = g^3$. Hence g^3 commutes with $\langle C, x \rangle = G$, so again $g^3 \in B \cap \mathfrak{Z}(G) = Z$.

Next, we show that C has rank 2. Suppose by way of contradiction that C has a subgroup Y of order 3 not contained in B. Then $D = ZY \cong Z \times Y$ is a normal subgroup of G, by Proposition 2.5 applied to $N = Z$. Furthermore, by Lemma 2.6, G/D is abelian and thus $G' \subseteq B \cap D = Z$. Now x acts like an element of order 3 on BD, an elementary abelian group of order 27. Viewing BD as a 3-dimensional vector space over $\mathrm{GF}(3)$, we can then view the action of x as a 3×3 matrix and consider its Jordan block structure. Of course, all eigenvalues of x are equal to $1 \in \mathrm{GF}(3)$. If there are at least two blocks, then $\mathfrak{C}_C(x) = \mathfrak{Z}(G)$ has rank at least 2, a contradiction. Thus there must be just one block of size 3, and this implies that the commutator $[C, x]$ has order ≥ 9, again a contradiction.

Thus C has rank 2 and, since C/Z has period 3, we conclude that C is abelian of type $(3, 3)$ or $(9, 3)$. In the former case, $|C| = 9$ so $|G| = 27$, contrary to the hypothesis of this lemma. Thus we must have C abelian of type $(9, 3)$, so $|C| = 27$ and $|G| = 81$. □

Since G is nonabelian and $G' \subseteq B$, there are two possibilities for G', namely $G' = Z$ or $G' = B$. We consider these two cases separately.

Claim 4. *If $G' = Z$, then G is the central product group of type (i).*

Proof. Since G has class 2 and G' has period 3, it follows from Lemma 2.8 and the previous claim that the map $g \mapsto g^3$ is a group homomorphism from G into Z. Furthermore, since C does not have period 3, the cube map is onto. The kernel N

is then a normal subgroup of G of index 3 and period 3. Clearly, N is nonabelian since otherwise $N \subseteq \mathfrak{C}_G(B) = C$ and this contradicts the fact that C has period 9. Thus N is isomorphic to the unique nonabelian group of order 27 and period 3.

Next, since C is abelian, the map $c \mapsto c^x c^{-1}$ is a homomorphism from C to $G' = Z$ with kernel $\mathfrak{C}_C(x) = \mathfrak{Z}(G)$. Thus $|\mathfrak{Z}(G)| = 9$, so $\mathfrak{Z}(G)$ is cyclic of order 9. It follows that $G = \mathfrak{Z}(G)N$ and this is the appropriate central product. □

Finally, we have

Claim 5. *If $G' = B$, then G is the group of type (ii).*

Proof. We first prove that B contains all elements of G of order 3. Indeed, if Y is a subgroup of G disjoint from Z, then Proposition 2.5 and Lemma 2.6 imply in turn that $ZY \triangleleft G$ and then that $ZY \supseteq G' = B$. Thus since $|ZY| = 9$, we conclude that $ZY = B$ and $Y \subseteq B$, as required.

Fix $a \in C$ of order 9. Then $a^3 \in Z$, so $Z = \langle a^3 \rangle$. By Claim 3 and the above, $x^3 \in Z$ and $x^3 \neq 1$. Thus, replacing x by x^{-1} if necessary, we can assume that $x^3 = a^3$. If $\langle a \rangle$ were normal in G, then its quotient group would have order 9 and hence be abelian. But this would then imply that $G' \subseteq \langle a \rangle \cap B = Z$, a contradiction. It therefore follows that $a^x a^{-1}$ is contained in $B = G'$ but not in $Z = \langle a^3 \rangle$. Setting $a^x a^{-1} = b$, we have $a^x = ab$, $b^3 = 1$ and $C = \langle a, b \rangle$. Furthermore, since b is not central in G, but $B/Z \subseteq \mathfrak{Z}(G/Z)$, we have $b^x b^{-1} = a^3$ or a^{-3}. In the latter case, $b^x = ba^{-3}$ and G is isomorphic to the group of type (ii).

On the other hand, if $b^x = ba^3$, then $a^{x^2} = (ab)^x = (ab)(ba^3) = a^4 b^2$ and hence $a^{x^2} a^x a = (a^4 b^2)(ab)(a) = a^6$. It then follows that $(xa)^3 = xaxaxa = x^3 a^{x^2} a^x a = a^3 a^6 = 1$, and this is a contradiction since we have shown that all elements of G of order 3 are contained in B. □

It is easy to see that if G is the type (ii) group of the previous claim, then we have $c^{x^2} c^x c = 1$ for all $c \in C$, and thus $(xc)^3 = x^3 c^{x^2} c^x c = x^3 \neq 1$. In particular, this group cannot be eliminated based upon the periodicity criterion. This completes the proof of the lemma. □

Next we consider MJD groups whose centers have rank 2. Here we obtain only one exception.

Lemma 2.10. *Let G be a nonabelian 3-group with MJD. If $\mathfrak{Z}(G)$ has rank 2, then G is the semidirect product $G = X \rtimes Y$, where X and Y are cyclic of order 9. In particular, $|G| = 81$ and G is the type (iii) group in Theorem 2.1.*

Proof. Let $W = Z_1 \times Z_2$ denote the elementary abelian subgroup of $\mathfrak{Z}(G)$ of order 9. Here, of course, $|Z_1| = |Z_2| = 3$.

Claim 1. *$G = X \rtimes Y$, where X and Y are cyclic of order ≥ 9 and where Y acts on X like a group of order 3.*

Proof. We first prove that W contains all elements of G of order 3. To this end, suppose that J is a subgroup of G of order 3 not contained in W. Then J is disjoint from Z_1 and Z_2, so Z_1J and Z_2J are both normal in G by Proposition 2.5. Furthermore, these two groups are distinct since J is not contained in W, so $J = Z_1J \cap Z_2J$ is normal in G. Hence J is central and $J \subseteq W$, contradiction.

Next, since W is not cyclic, G/W is abelian by Lemma 2.6. Thus G has class 2 and $G' \subseteq W$ has period 3. Lemma 2.8 now implies that the map $\theta\colon g \mapsto g^3$ is a homomorphism from G to $\mathfrak{Z}(G)$. Indeed, since $\ker\theta = W$ has order 9 and since $|G : \mathfrak{Z}(G)| \geq 9$ when G is nonabelian, we conclude that θ is onto $\mathfrak{Z}(G)$ and that $G/\mathfrak{Z}(G)$ is elementary abelian of order 9. The latter implies that $G = \langle \mathfrak{Z}(G), g_1, g_2\rangle$ for two group elements g_1 and g_2, and hence G' is generated by the central commutator $z = [g_1, g_2]$ of order 3. In other words, G' is central of order 3.

It now follows easily from the fundamental theorem of abelian groups that $\mathfrak{Z}(G) = X_1 \times Y_1$, where X_1 and Y_1 are cyclic and $X_1 \supseteq G'$. Finally, since θ is onto, there exist cyclic subgroups X and Y of G with $|X : X_1| = 3$ and $|Y : Y_1| = 3$. In particular, $X \supseteq G'$, so $X \triangleleft G$, and since $X \cap Y = 1$, we have $G = X \rtimes Y$. □

It remains to find $|X|$ and $|Y|$. We start with

Claim 2. $|X| = 9$.

Proof. We already know that $|X| \geq 9$. If $|X| \geq 27$, let X_2 be the unique subgroup of X of index 9. Then $|X_2| \geq 3$, so $X_2 \supseteq G'$. Furthermore, X_2 is central in G and hence $A = X_2Y$ is a normal abelian subgroup of G. Note that $G/A \cong X/X_2$ is cyclic of order 9 and that $A \supseteq Y$, a nonnormal subgroup of G. Since $|G'| = 3$, this contradicts Proposition 2.4, and the claim is proved. □

We now know that $|X| = 9$ and we move on to the cyclic group Y. An argument similar to the above can show that $|Y| \leq 27$. Instead, we use a variant of [HPW, Lemma 24] to get the sharper result.

Claim 3. $|Y| = 9$.

Proof. Let $X = \langle x\rangle$ and write $z = x^3 \in \mathfrak{Z}(G)$. Then, by replacing y by y^{-1} if necessary, we can assume that $Y = \langle y\rangle$ with $x^y = xz$. Of course, $y^3 \in \mathfrak{Z}(G)$. Now suppose, by way of contradiction, that $|Y| \geq 27$ and let $W = \langle w\rangle$ be the unique subgroup of Y of order 9. Then $W \subseteq \mathfrak{Z}(G)$, so $C = XW$ is abelian, and y acts on C as an element of order 3. If $\gamma \in \mathbb{Z}[C]$, let us write $N(\gamma) = \gamma\gamma^y\gamma^{y^2}$. Since C is abelian, $N\colon \mathbb{Z}[C] \to \mathbb{Z}[C]$ is a multiplicative homomorphism and we can write the three factors in any order.

If $t \in W$, then

$$\begin{aligned} N(x-t) &= (x-t)(x-t)^y(x-t)^{y^2} = (x-t)(xz-t)(xz^2-t) \\ &= x^3 - x^2t(1+z+z^2) + xt^2(1+z+z^2) - t^3 \end{aligned}$$

and hence

$$(z-1)N(x-t) = (z-1)(x^3-t^3) = (z-1)(z-t^3).$$

In particular, if $\alpha = (z-1)(x-w)(x-w^2)(w^3-1)$, then

$$\begin{aligned} N(\alpha) &= (z-1)\cdot(z-1)N(x-w)\cdot(z-1)N(x-w^2)\cdot(w^3-1)^3 \\ &= (z-1)^2\cdot(z-1)(z-w^3)(z-w^6)\cdot(w^3-1)^3 = 0 \end{aligned}$$

since

$$\begin{aligned} (z-1)(z-w^3)(z-w^6) &= z^3 - z^2(1+w^3+w^6) + z(1+w^3+w^6) - 1 \\ &= (z-z^2)(1+w^3+w^6) \end{aligned}$$

is annihilated by w^3-1.

It follows that $(y\alpha)^3 = y^3N(\alpha) = 0$ and hence, by Theorem 1.1(ii), $y\alpha\widehat{U}/3 \in \mathbb{Z}[G]$, where $U = \langle u\rangle$ is the central subgroup of order 3 generated by $u = zw^3$. Thus $\alpha\widehat{U}/3 \in \mathbb{Z}[G]$, and by considering the central group elements in the support of this element, we see that

$$(z-1)w^3(w^3-1)(1+u+u^2)/3 \in \mathbb{Z}[G].$$

Canceling the w^3 factor and considering those group elements in U, we conclude that

$$(1+u)(1+u+u^2)/3 = (1+u+u^2)(2/3) \in \mathbb{Z}[G],$$

certainly a contradiction. Thus, $|Y| \le 9$, as required. □

The latter two claims now clearly yield the result. □

If G is a nonabelian 3-group with MJD then, by Lemma 2.7, $\mathfrak{Z}(G)$ has rank ≤ 2. Since Lemmas 2.9 and 2.10 handle the rank 1 and rank 2 cases, respectively, Theorem 2.1 is now proved.

3. Groups of order 3^3

In this section, we study the two nonabelian groups of order 3^3. The main result, Theorem 3.5 asserts that both of these groups satisfy MJD.

Let G be either of the two nonabelian groups of order $3^3 = 27$, so that $\mathfrak{Z}(G)$, the center of G, is cyclic of order 3 with generator z. We are concerned with the integral group ring $\mathbb{Z}[G]$ and the rational group algebra $\mathbb{Q}[G]$. Let ω be a primitive complex cube root of unity and define $F = \mathbb{Q}[\omega]$ and $R = \mathbb{Z}[\omega]$. Then, as is well known, R is the ring of algebraic integers in the quadratic field F, it is a Euclidean domain, and it has precisely six units, namely ± 1, $\pm\omega$ and $\pm\omega^2$. We fix this notation throughout the following few results.

Lemma 3.1. *Let G be as above. Then*

i. $\mathbb{Q}[G/\mathfrak{Z}(G)] \cong \mathbb{Q} \oplus F \oplus F \oplus F \oplus F$.

ii. $\mathbb{Q}[G] \cong \mathbb{Q}[G/\mathfrak{Z}(G)] \oplus \mathrm{M}_3(F)$*, and we let* $\theta\colon \mathbb{Q}[G] \to \mathrm{M}_3(F)$ *denote the natural projection.*
iii. We can assume that $\theta(z) = \omega I$ *and that* $\theta(\mathbb{Z}[G]) \subseteq \mathrm{M}_3(R)$*.*

Proof. Part (i) is clear since $G/\mathfrak{Z}(G)$ is elementary abelian of order 9. For part (ii), we know that $\mathbb{Q}[G/\mathfrak{Z}(G)]$ is isomorphic to a ring direct summand of $\mathbb{Q}[G]$, and we proceed to exhibit the map θ in a concrete manner. To start with, note that G has a normal elementary abelian subgroup $A = \langle z\rangle \times \langle x\rangle$ of order 9, and that $G = \langle A, y\rangle$ where $x^y = xz$ and $y^3 = z^i$ for $i = 0$ or 1. Next, we define $\theta\colon G \to \mathrm{M}_3(R)$ by $\theta(z) = \omega I$, where I is the identity matrix, $\theta(x) = \mathrm{diag}(1, \omega, \omega^2)$ and

$$\theta(y) = \begin{pmatrix} 0 & 0 & \omega^i \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then it is easy to verify that θ is a representation of G with $\theta(\mathbb{Q}[G]) = \mathrm{M}_3(F)$, and dimension considerations imply that $\mathbb{Q}[G] \cong \mathbb{Q}[G/\mathfrak{Z}(G)] \oplus \mathrm{M}_3(F)$. □

We continue to use $\theta\colon \mathbb{Q}[G] \to \mathrm{M}_3(F)$ for the natural projection given above. In addition, we let $\mathrm{tr}\colon \mathrm{M}_3(F) \to F$ and $\det\colon \mathrm{M}_3(F) \to F$ denote the usual matrix trace and determinant.

Lemma 3.2. *If* $g \in G \setminus \mathfrak{Z}(G)$*, then* $\mathrm{tr}\,\theta(g) = 0$*. Thus* $\mathrm{tr}\,\theta(\mathbb{Z}[G]) \subseteq 3R$*.*

Proof. This is a standard argument. If $g \in G \setminus \mathfrak{Z}(G)$, then there exists $h \in G$ not commuting with g. Since $G/\mathfrak{Z}(G)$ is abelian, it follows that $h^{-1}gh = z^j g$ for $j = 1$ or 2, and hence $\theta(h)^{-1}\theta(g)\theta(h) = \theta(z^j g) = \omega^j\theta(g)$. Using the fact that similar matrices have the same trace, we get $\mathrm{tr}\,\theta(g) = \mathrm{tr}\,\theta(h)^{-1}\theta(g)\theta(h) = \mathrm{tr}\,\omega^j\theta(g) = \omega^j\,\mathrm{tr}\,\theta(g)$, and hence $\mathrm{tr}\,\theta(g) = 0$ since $\omega^j \neq 1$. In particular, $\mathrm{tr}\,\theta(\mathbb{Z}[G]) = \mathrm{tr}\,\theta(\mathbb{Z}[\mathfrak{Z}(G)])$. But $\mathrm{tr}\,\theta(z^i) = \mathrm{tr}\,\omega^i I = 3\omega^i$, so we conclude that $\mathrm{tr}\,\theta(\mathbb{Z}[G]) \subseteq 3R$. □

Of course, $F/\mathbb{Q}$ is Galois with group $\{1, \sigma\}$, where σ is defined by $\omega^\sigma = \omega^2$. Let $N\colon F \to \mathbb{Q}$ denote the usual Galois norm map given by $N(\alpha) = \alpha\alpha^\sigma$.

Lemma 3.3. *If* $N\colon F \to \mathbb{Q}$ *is the norm map, then* $N(R) \subseteq \mathbb{Z}$ *and* $N(1-\omega) = 3$*. Furthermore,* R *has precisely six units, namely* ± 1*,* $\pm\omega$ *and* $\pm\omega^2$*, and no two of these are congruent modulo* $3R$*.*

Proof. The inclusion $N(R) \subseteq \mathbb{Z}$ follows from the fact that R is integral over $\mathbb{Z}$. Furthermore, ω is a root of the polynomial $f(\zeta) = \zeta^2 + \zeta + 1 = (\zeta - \omega)(\zeta - \omega^2)$, so $N(1-\omega) = (1-\omega)(1-\omega^2) = f(1) = 3$. Finally, we know that the units of R are ± 1, $\pm\omega$ and $\pm\omega^2$. If a and b are two such units with $a - b \in 3R$, then by multiplying by a^{-1}, we can assume that $a = 1$. Clearly $b \neq -1$, and $b \neq \pm\omega$ or $\pm\omega^2$ since $\{1, \omega\}$ and $\{1, \omega^2\}$ are both $\mathbb{Z}$-bases for R. Thus $b = 1 = a$, as required. □

Recall that a matrix is said to be unipotent if all its eigenvalues are equal to 1. Furthermore, an element $\alpha \in \mathbb{Q}[G]$ is unipotent, if its minimal polynomial over $\mathbb{Q}$ has all roots equal to 1. We now come to the first key result of this section.

Proposition 3.4. *Let u be a unit in $\mathbb{Z}[G]$ and suppose that $\theta(u)$ is a unipotent matrix. Then u maps to 1 in $\mathbb{Q}[G/\mathfrak{Z}(G)]$ and hence u is unipotent in $\mathbb{Q}[G]$.*

Proof. Let $\bar{\ }\colon \mathbb{Z}[G] \to \mathbb{Z}[G/\mathfrak{Z}(G)]$ denote the natural epimorphism, so that $\bar{u}$ is a unit in $\mathbb{Z}[G/\mathfrak{Z}(G)]$. Since $G/\mathfrak{Z}(G)$ is an elementary abelian 3-group, Higman's theorem [Hi, Theorem 6] implies that all units in $\mathbb{Z}[G/\mathfrak{Z}(G)]$ are trivial. Thus we can write $u = \pm g + \beta$, where $g \in G$ and $\bar{\beta} = 0$. Indeed, since g can be chosen from a set of coset representatives of $\mathfrak{Z}(G)$ in G, we can assume that $g \in \mathfrak{Z}(G)$ implies $g = 1$. Our goal is to show that $g = 1$ and that only the plus sign can occur in $u = \pm g + \beta$.

Note that the kernel of $\bar{\ }$ is the principal ideal generated by $1-z$, so $\beta = (1-z)\alpha$ for some $\alpha \in \mathbb{Z}[G]$. Now, by hypothesis, $\theta(u) = \theta(\pm g + (1-z)\alpha)$ is unipotent and hence this 3×3 matrix has trace equal to 3. In other words,

$$\pm \operatorname{tr}\theta(g) + (1-\omega)\operatorname{tr}\theta(\alpha) = 3.$$

Furthermore, by Lemma 3.2, we know that $\operatorname{tr}\theta(\alpha) = 3r$ for some $r \in R$. Thus

$$\pm \operatorname{tr}\theta(g) + (1-\omega)3r = 3.$$

Now, if $g \in G \setminus \mathfrak{Z}(G)$, then $\operatorname{tr}\theta(g) = 0$ by Lemma 3.2. Thus the above equation yields $1 = (1-\omega)r$ and, by applying the norm map $N\colon R \to \mathbb{Z}$, Lemma 3.3 yields $1 = N(1) = N(1-\omega)\cdot N(r) \in 3\mathbb{Z}$, certainly a contradiction. Thus we must have $g \in \mathfrak{Z}(G)$ and hence $g = 1$. Next, if the minus sign occurs in the above displayed equation, then $\operatorname{tr}\theta(g) = \operatorname{tr}\theta(1) = 3$ so $-3 + (1-\omega)3r = 3$ and $2 = (1-\omega)r$. By applying the norm map, Lemma 3.3 yields $4 = N(2) = N(1-\omega)\cdot N(r) \in 3\mathbb{Z}$, and this is again a contradiction. Thus only the plus sign can occur and the proposition is proved. □

If u is a unit in $\mathbb{Z}[G]$, let us write $u = u_s u_1$ for its multiplicative Jordan decomposition in $\mathbb{Q}[G]$. Here, of course, u_s is its semisimple part and u_1 is a unipotent unit commuting with u_s. We can now prove

Theorem 3.5. *Let G be a nonabelian group of order 27 with $\mathfrak{Z}(G) = \langle z \rangle$, and let u be a unit in the integral group ring $\mathbb{Z}[G]$. Then either u is semisimple and $u = u_s$, or $u_s = (-z)^i$ for some $i = 0, 1, \ldots, 5$. In either case, u_s is a unit in $\mathbb{Z}[G]$, and hence $\mathbb{Z}[G]$ satisfies the multiplicative Jordan decomposition property.*

Proof. Suppose u is not semisimple. Then, since $F/\mathbb{Q}$ is a separable field extension, it follows from Lemma 3.1(i)(ii) that the matrix $\theta(u) \in \mathrm{M}_3(F)$ is not semisimple. In particular, its characteristic polynomial $f(\zeta) \in F[\zeta]$ is a monic polynomial of degree 3 with multiple roots, and say these roots are λ, λ and μ. Now $f(\zeta)$ must be divisible by the square of the minimal polynomial $g(\zeta)$ satisfied by λ over F, so $\deg f = 3$ implies that $\deg g = 1$ and $\lambda \in F$. Clearly, μ is now also contained in F. Furthermore, since $\theta(u) \in \mathrm{M}_3(R)$, it follows that $f(\zeta) \in R[\zeta]$ is monic and, since R is integrally closed in F, the roots must all be contained in R. We are given that u is a unit in $\mathbb{Z}[G]$, so $\theta(u)$ is a unit in $\mathrm{M}_3(R)$, and hence $\det \theta(u)$ is a unit in R. But $\lambda^2\mu = \det \theta(u)$, so λ and μ are necessarily units in R. Note also that $\operatorname{tr} \theta(u) = 2\lambda + \mu \in 3R$ by Lemma 3.2, so $\mu - \lambda = (2\lambda + \mu) - 3\lambda \in 3R$ and it follows from Lemma 3.3 that $\mu = \lambda$.

We now know that all eigenvalues of $\theta(u)$ are equal to λ, and that λ is a unit in R. Thus $\lambda = (-\omega)^i$ for some $i = 0, 1, \ldots, 5$. Since z is central and $\theta(z) = \omega I$, we see that $v = u(-z)^{-i}$ is a unit of $\mathbb{Z}[G]$ and that all eigenvalues of $\theta(v)$ are equal to 1. Thus $\theta(v)$ is unipotent, and hence so is v by Proposition 3.4. In other words, $u = v(-z)^i$ is a product of commuting units with $(-z)^i$ semisimple and with v unipotent. It therefore follows from uniqueness of the multiplicative Jordan decomposition that $u_s = (-z)^i$, as required. □

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