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MULTIPLICATIVE JORDAN DECOMPOSITION IN GROUP RINGS OF 2, 3-GROUPS

CHIA-HSIN LIU

Department of Mathematics
National Taiwan Normal University
Taipei, Taiwan, R.O.C.
chliu@math.ntnu.edu.tw

D. S. PASSMAN

Department of Mathematics
University of Wisconsin-Madison
Madison, Wisconsin 53706, USA
passman@math.wisc.edu

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In this paper, we essentially finish the classification of those finite 2, 3-groups G having integral group rings with the multiplicative Jordan decomposition (MJD) property. If G is abelian or a Hamiltonian 2-group, then it is clear that $\mathbb{Z}[G]$ satisfies MJD. Thus, we need only consider the nonabelian case. Recall that the 2-groups with MJD were completely determined by Hales, Passi and Wilson, while the corresponding 3-groups were almost completely determined by the present authors. Thus, we are concerned here, for the most part, with groups whose order is divisible by 6. As it turns out, there are precisely three nonabelian 2, 3-groups, of order divisible by 6, with $\mathbb{Z}[G]$ satisfying MJD. These have orders 6, 12, and 24. In view of another result of Hales, Passi and Wilson, this completes a significant portion of the classification of all finite groups with MJD.

Keywords: Integral group ring; multiplicative Jordan decomposition; 2, 3-group.

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1. Introduction

Let $\mathbb{Q}[G]$ denote the rational group algebra of the finite group G. Since $\mathbb{Q}$ is a perfect field, every element a of $\mathbb{Q}[G]$ has a unique additive Jordan decomposition $a = a_s + a_n$, where a_s is a semisimple element and where a_n commutes with a_s and is nilpotent. If a is a unit, then a_s is also invertible and $a = a_s(1 + a_s^{-1}a_n)$ is a product of a semisimple unit a_s and a commuting unipotent unit $a_u = 1 + a_s^{-1}a_n$. This is the unique multiplicative Jordan decomposition of a. Following [2] and [5], we say that $\mathbb{Z}[G]$ has the multiplicative Jordan decomposition property (MJD) if for every unit a of $\mathbb{Z}[G]$, its semisimple and unipotent parts are both contained in $\mathbb{Z}[G]$. For simplicity, we say that G satisfies MJD if its integral group ring $\mathbb{Z}[G]$ has this property.

If G is abelian or a Hamiltonian 2-group, then every element of $\mathbb{Q}[G]$ is semisimple. Thus, every unit a of $\mathbb{Z}[G]$ is equal to its semisimple part and consequently $\mathbb{Z}[G]$ trivially satisfies MJD. In the non-Dedekind case, it appears that the MJD property is relatively rare. Indeed, the papers [2] and [5] have shown that $\mathbb{Z}[G]$ and $\mathbb{Q}[G]$ must be quite restrictive. For example, we have the following, with part (i) from [2, Theorem 4.1] and part (ii) from [5, Corollary 9].

Theorem 1.1. *Let G have the multiplicative Jordan decomposition property.*

(i) *If the matrix ring $\mathrm{M}_n(D)$, over the division ring D, is a Wedderburn component of $\mathbb{Q}[G]$, then $n \leq 3$.*
(ii) *If η is a nilpotent element of $\mathbb{Z}[G]$ and e is a central idempotent of $\mathbb{Q}[G]$, then $\eta e \in \mathbb{Z}[G]$.*

Using this and numerous clever arguments, paper [5] was able to determine all nonabelian 2-groups that satisfy MJD. Specifically, these are the two nonabelian groups of order 8, five groups of order 16, four groups of order 32, and only the Hamiltonian groups of larger order. On the other hand, in paper [6] we were able to build on the work of [5], using variants of many of the same arguments, to essentially determine all nonabelian 3-groups satisfying MJD. These are the two nonabelian groups of order 3^3 and at most three groups of order 3^4. As we will see, the computations of the next section eliminate two of these possibilities, sharpening [6, Theorems 2.1 and 3.5] to yield

Theorem 1.2. *Let G be a finite nonabelian 3-group. If $|G| = 3^3$, then $\mathbb{Z}[G]$ has the MJD property. On the other hand, if G has order $\geq 3^4$ and $\mathbb{Z}[G]$ satisfies MJD, then G can only be one possible group of order 3^4, namely the central product of a cyclic group of order 9 with the nonabelian group of order 27 and period 3.*

We still do not know whether the latter group has MJD. Indeed, this is the only open group among all the $2,3$-groups with MJD since our main result asserts

Theorem 1.3. *Let G be a finite nonabelian $2,3$-group with order divisible by 6. Then $\mathbb{Z}[G]$ satisfies MJD if and only if*

(i) *$G = \mathrm{Sym}_3$, the symmetric group of degree 3,*
(ii) *$G = \langle x, y \mid x^3 = 1,\, y^4 = 1,\, y^{-1}xy = x^{-1}\rangle$, a group of order 12, or*
(iii) *$G = Q_8 \times C_3$, the direct product of the quaternion group of order 8 with the cyclic group of order 3.*

This will be proved in Sec. 3. As is apparent from [5, Theorem 29], the latter two results complete a significant portion of the characterization of those finite groups G with $\mathbb{Z}[G]$ satisfying MJD.

2. Negative Computations

In this section, we begin our work by showing that several concrete groups fail to have the MJD property. As usual, if X is a subset of G, we write $\widehat{X}$ for the sum of the members of X in $\mathbb{Z}[G]$. Of course, if H is a subgroup of G and if $h \in H$, then $h\widehat{H} = \widehat{H}\,h = \widehat{H}$. In particular, $e_H = \widehat{H}/|H|$ is the principal idempotent in $\mathbb{Q}[H]$ determined by H. If $H \triangleleft G$, then e_H is central in $\mathbb{Q}[G]$ and, as is well-known, $e_H\mathbb{Q}[G]$ is naturally isomorphic to $\mathbb{Q}[G/H]$.

Lemma 2.1. *Let G be the group of order 81 given by*

$$G = \langle x, y, z \mid x^9 = 1,\, y^3 = 1,\, xy = yx,\, x^z = xy,\, y^z = yx^{-3},\, z^3 = x^3\rangle.$$

Then $\mathbb{Z}[G]$ does not have MJD.

Proof. Note that $A = \langle x, y\rangle$ is an abelian subgroup of G and that z acts on A and hence on $\mathbb{Z}[A]$ as an automorphism of order 3. If $\alpha \in \mathbb{Z}[A]$, define the norm of α by $N(\alpha) = \alpha\alpha^z\alpha^{z^2}$. Obviously, N is multiplicative and the ordering of the factors in N does not matter. In particular, in $\mathbb{Z}[G]$ we have $(z\alpha)^3 = z^3\alpha^{z^2}\alpha^z\alpha = z^3N(\alpha)$.

Now set $\alpha = (1-x)(1-y) \in \mathbb{Z}[A]$. Then

$$\begin{aligned}N(1-x) &= (1-x)(1-x^z)(1-x^{z^2}) = (1-x)(1-xy)(1-xy^2x^{-3})\\ &= (1-x)(1-xy)[1-xy^2-xy^2x^{-3}(1-x^3)]\\ &= (x^2-x)\widehat{Y} + (1-x^3)\beta\end{aligned}$$

where $Y = \langle y\rangle$ and $\beta \in \mathbb{Z}[A]$. Furthermore, we have

$$\begin{aligned}N(1-y) &= (1-y)(1-y^z)(1-y^{z^2})\\ &= (1-y)(1-yx^{-3})(1-yx^{-6}) = (y^2-y)\widehat{W}\end{aligned}$$

where $W = \langle x^3\rangle$. Since $\widehat{W}$ annihilates $1-x^3$, and y^2-y annihilates $\widehat{Y}$, we conclude that

$$N(\alpha) = N(1-x)\,N(1-y) = 0.$$

Finally, if $\eta = z\alpha$, then $\eta^3 = z^3\,N(\alpha) = 0$. Furthermore, if $e = \widehat{W}/3 \in \mathbb{Q}[G]$ is the central idempotent determined by $W = \langle x^3\rangle$, then

$$\eta e = z(1-x)(1-y)(1+x^3+x^6)/3$$

has z-coefficient equal to $1/3$. Thus $\eta e \notin \mathbb{Z}[G]$ and Theorem 1.1(ii) implies that $\mathbb{Z}[G]$ does not satisfy MJD. □

The next group is

Lemma 2.2. *Let G be the 3-group of order 81 given by*

$$G = \langle x, y \mid x^9 = 1,\, y^9 = 1,\, y^{-1}xy = x^4\rangle.$$

Then $\mathbb{Z}[G]$ does not have MJD.

Proof. Note that $G' = \langle x^3 \rangle$ is central and that $Z = \mathfrak{Z}(G) = \langle x^3 \rangle \times \langle y^3 \rangle$ is abelian of type $(3, 3)$. Furthermore, we have $xy = yx^4$, and then multiplying by $x^{-3} \in Z$ yields $yx = x^{-2}y$.

Define $\alpha = 1 + x + x^2 \in \mathbb{Z}[G]$ and set $\beta = (1 - x^2y)(1 - xy)(1 - y)$. Since $(x^2y)(xy) = x^2(yx)y = x^2(x^{-2}y)y = y^2$, we have $\beta = 1 - \alpha y + \alpha y^2 - y^3$. Now let $\eta = (1 - x^3)x^2\beta \in \mathbb{Z}[G]$. We will show below that $\eta^3 = 0$. Assuming this is the case, let $e = (1 + y^3 + y^6)/3$ be the central idempotent in $\mathbb{Q}[G]$ determined by the central subgroup $\langle y^3 \rangle$ of order 3. Then $(1 - y^3)e = 0$, so

$$\eta e = (1 - x^3)x^2\alpha(y^2 - y)(1 + y^3 + y^6)/3$$

and this has its x^2y^2-coefficient equal to $1/3$. Thus $\eta e \notin \mathbb{Z}[G]$ and we conclude from Theorem 1.1(ii) that $\mathbb{Z}[G]$ does not satisfy MJD.

It remains to prove that $\eta^3 = 0$. This can be verified by hand via a tedious computation or easily by computer by first determining the 81×81 matrices for x and y that define the regular representation for $\mathbb{Z}[G]$. We take a third approach using the irreducible representations of the complex group algebra $\mathbb{C}[G]$.

Let $\mathfrak{X}$ be such a representation with character χ. If $\deg \mathfrak{X} = 1$, then $x^3 \in \ker_G(\mathfrak{X})$, so $\mathfrak{X}(\eta) = 0$. Thus it suffices to assume that $\deg \mathfrak{X} = 3$. In this case, we know that $x^3 \notin \ker_G(\mathfrak{X})$. On the other hand, $\mathfrak{X}$ cannot be faithful on the center of G which is abelian of type $(3, 3)$. Thus we must have $x^{3k}y^3 \in \ker_G(\mathfrak{X})$ for some $k = 0, 1$, or 2. In other words, $\mathfrak{X}$ corresponds to an irreducible representation $\overline{\mathfrak{X}}$ of $\overline{G} = G/\langle x^{3k}y^3 \rangle$, a nonabelian group of order 27 with center $\overline{Z} = Z/\langle x^{3k}y^3 \rangle$. Basic properties of such representations can be found in [6, Lemmas 3.1 and 3.2]. In particular, the latter reference implies that the corresponding character $\overline{\chi}$ vanishes off the center of $\overline{G}$. Thus, in G, we see that $\chi(x^iy^j) = 0$ unless both i and j are divisible by 3.

Let $\overline{x}$ and $\overline{y}$ denote the images of elements x and y, respectively, in $\overline{G}$. Then by [6, Lemma 2.8], we see that $(\overline{x}^k\overline{y})^3 = 1$. Thus since $\overline{\mathfrak{X}}$ is induced from a linear representation of $\langle \overline{x} \rangle$, we see that $\overline{\mathfrak{X}}$ restricted to $\langle \overline{x}^k\overline{y} \rangle$ is the regular representation of this cyclic group of order 3. (This also follows from our knowledge of $\overline{\chi}$ restricted to this subgroup.) In particular, $\mathfrak{X}(x^ky) = \overline{\mathfrak{X}}(\overline{x}^k\overline{y})$ has an eigenvalue equal to 1, so $\det \mathfrak{X}(1 - x^ky) = 0$. Hence $\det \mathfrak{X}(\beta) = 0$ and $\det \mathfrak{X}(\eta) = 0$. In addition, since

$$\eta = (1 - x^3)x^2(1 - \alpha y + \alpha y^2 - y^3),$$

it is clear that $\chi(\eta) = 0$. We now compute $\chi(\eta^2)$.

Note that

$$\eta^2 = (1 - x^3)^2x^2(1 - \alpha y + \alpha y^2 - y^3)x^2(1 - \alpha y + \alpha y^2 - y^3)$$

and $\chi(x^iy^j) = 0$ if 3 does not divide j. Furthermore, $\chi\big((1 - x^3)^2x^4\big) = 0$ and, since $\mathfrak{X}(y^3)$ is a scalar matrix, we have

$$\chi\big((1 - x^3)^2x^2(1 - y^3)x^2(1 - y^3)\big) = 0.$$

Thus since $\chi(ab) = \chi(ba)$, it follows that

$$\chi(\eta^2) = -2\chi\big((1-x^3)^2x^2\alpha y x^2\alpha y^2\big).$$

Now $yxy^{-1} = x^{-2} = x^7$ and $(1-x^3)$ annihilates $\widehat{X} = \widehat{\langle x\rangle} = 1 + x + \cdots + x^8$, so

$$\begin{aligned}(1-x^3)^2x^2\alpha y x^2\alpha y^2 &= (1-x^3)^2x^2\alpha \cdot y(x^2\alpha)y^{-1}\cdot y^3\\ &= (1-x^3)^2x^2\cdot(1+x+x^2)(x^5+x^3+x)\cdot y^3\\ &= (1-x^3)^2x^2\cdot(\widehat{X}+x^3+x^5-1-x^8)\cdot y^3\\ &= (1-x^3)^3x^2\cdot(x^5-1)\cdot y^3 = (1-x^3)^3\cdot(x^7-x^2)\cdot y^3\end{aligned}$$

and hence $\chi(\eta^2) = 0$.

Finally, since $\det\mathfrak{X}(\eta) = 0$ and $\operatorname{tr}\mathfrak{X}(\eta) = \chi(\eta) = 0$, we see that the characteristic roots of $\mathfrak{X}(\eta)$ are 0, λ, $-\lambda$ for some $\lambda \in \mathbb{C}$. Thus the characteristic roots of $\mathfrak{X}(\eta^2)$ are 0, λ^2, λ^2 so $0 = \chi(\eta^2) = 2\lambda^2$ and hence $\lambda = 0$. In other words, all three characteristic roots of $\mathfrak{X}(\eta)$ are 0, so $\mathfrak{X}(\eta^3) = \mathfrak{X}(\eta)^3 = 0$. Since $\mathbb{C}[G]$ is semisimple and $\mathfrak{X}$ is arbitrary, we conclude that $\eta^3 = 0$, as required. ◻

Theorem 1.2 now follows from [6, Theorems 2.1 and 3.5] and the preceding two results. The next computation concerns certain groups of order $2^n \cdot 3$ and is due to Hales and Passi in [4]. We thank Profs. Hales and Passi for allowing us to include it here.

Lemma 2.3. *Let G be the group of order $2^n\cdot 3$ given by*

$$G = \langle x, y \mid x^3 = 1,\ y^{2^n} = 1,\ y^{-1}xy = x^{-1}\rangle.$$

If $n \geq 3$, then $\mathbb{Z}[G]$ does not have MJD.

Proof. For each $0 \le i \le n$, set $Y_i = \langle y^{2^i}\rangle$. Thus

$$Y_0 = \langle y\rangle \supset Y_1 \supset Y_2 \supset \cdots \supset Y_n = \langle 1\rangle,$$

Y_1 is central in G and $|Y_i| = 2^{n-i}$. Define $\alpha = (x - x^{-1})(1 + xy + y^3) \in \mathbb{Z}[G]$ and set $\eta = \alpha(1+y^2)\widehat{Y}_3$. Note that Y_3 exists since $n \geq 3$.

We will show below that $\eta^2 = 0$. Assuming this is the case, let $e = \widehat{Y}_2/2^{n-2}$ be the central idempotent determined by $Y_2 = \langle y^4\rangle$. Since $\widehat{Y}_3\widehat{Y}_2 = |Y_3|\widehat{Y}_2 = 2^{n-3}\widehat{Y}_2$, we see that

$$\eta e = (x - x^{-1})(1 + xy + y^3)(1 + y^2)\widehat{Y}_2/2.$$

Furthermore, $(1+y^2)\widehat{Y}_2 = \widehat{Y}_1$ and $y^3\widehat{Y}_1 = y\widehat{Y}_1$ since $Y_1 = \langle y^2\rangle$. Thus

$$\eta e = (x - x^{-1})(1 + xy + y)\widehat{Y}_1/2$$

and this has x-coefficient equal to $1/2$. Thus $\eta e \notin \mathbb{Z}[G]$ and we conclude from Theorem 1.1(ii) that $\mathbb{Z}[G]$ does not have MJD.

It remains to prove that $\eta^2 = 0$. To start with, since y^2 is central, we have

$$\begin{aligned}(xy + y^3)^2 &= (x + y^2)y(x + y^2)y = (x + y^2)(x + y^2)^y y^2 \\ &= (x + y^2)(x^{-1} + y^2)y^2 = (1 + (x + x^{-1})y^2 + y^4)y^2.\end{aligned}$$

Thus since $(x - x^{-1})(1 + x + x^{-1}) = 0$ and $y(x - x^{-1}) = -(x - x^{-1})y$, we have

$$\begin{aligned}\alpha^2 &= (x - x^{-1})(1 + xy + y^3)(x - x^{-1})(1 + xy + y^3) \\ &= (x - x^{-1})^2(1 - xy - y^3)(1 + xy + y^3) \\ &= (x - x^{-1})^2(1 - (xy + y^3)^2) \\ &= (x - x^{-1})^2(1 - y^2 - (x + x^{-1})y^4 - y^6) \\ &= (x - x^{-1})^2(1 - y^2 + y^4 - y^6).\end{aligned}$$

Finally, note that

$$(1 - y^2 + y^4 - y^6)(1 + y^2)\widehat{Y}_3 = (1 - y^8)\widehat{Y}_3 = 0$$

since $Y_3 = \langle y^8 \rangle$. Thus we conclude that

$$\begin{aligned}\eta^2 &= \alpha^2(1 + y^2)^2\widehat{Y}_3^2 \\ &= (x - x^{-1})^2 \cdot (1 - y^2 + y^4 - y^6)(1 + y^2)\widehat{Y}_3 \cdot (1 + y^2)\widehat{Y}_3 = 0,\end{aligned}$$

as required. □

We need one more negative example.

Lemma 2.4. *Let $V = \{1, u, v, w\}$ be a four group and let $G = V\langle x \rangle$, where x has finite order divisible by 3 and acts on V by $u^x = v$, $v^x = w$ and $w^x = u$. Then G does not have MJD.*

Proof. Set $c = x - x^u$. Then $c^u = x^u - x = -c$ so $uc + cu = 0$. Next observe that $x^{-1}ux = v$ yields $ux = xv$. Similarly, $vx = xw$ and $wx = xu$. Thus $x^u = uxu = xvu = xw$ so $c = x - x^u = x - xw = x(1 - w)$. Next, we have

$$c^2 = x(1 - w)x(1 - w) = x^2(1 - u)(1 - w),$$

so clearly $c^2 \notin 2\mathbb{Z}[G]$. Furthermore,

$$\begin{aligned}c^3 &= c^2 \cdot c = x^2(1 - u)(1 - w) \cdot x(1 - w) \\ &= x^3(1 - v)(1 - u)(1 - w) = 0\end{aligned}$$

since

$$(1 - v)(1 - u) = 1 + w - (u + v) = 1 + w - u(1 + w) = (1 - u)(1 + w)$$

and $(1 + w)(1 - w) = 1 - w^2 = 0$.

Now define $a = u + c$. Since $u^2 = 1$ and $uc + cu = 0$, we have $a^2 = 1 + c^2 = 1 + d$, where for convenience we set $d = c^2$. Then $d^2 = 0$ since $c^3 = 0$, so $a^{-2} = 1 - d$ and $a^{-1} = a \cdot a^{-2} = a(1 - d)$. Consequently, a is a unit in $\mathbb{Z}[G]$.

Suppose $a = a_s a_u = s(1+r)$ is the product of its semisimple and unipotent part, so that r is nilpotent. Since s and $1 + r$ commute, we have $1 + d = a^2 = s^2(1 + r)^2$. By uniqueness, $s^2 = 1$ and $1 + d = (1 + r)^2 = 1 + 2r + r^2$. In particular, $2r + r^2 = d$.

Finally, we have $2r(1 + r/2) = d$, and note that $1 + r/2$ is a unit in $\mathbb{Q}[G]$ that commutes with the left-hand side and hence with d. We then have $2r = d(1+r/2)^{-1}$ and since $d^2 = 0$, we conclude that $r^2 = 0$. It follows that $c^2 = d = 2r + r^2 = 2r$. But $c^2 \notin 2\,\mathbb{Z}[G]$, so $1 + r \notin \mathbb{Z}[G]$ and $\mathbb{Z}[G]$ does not satisfy MJD. □

3. The Main Result

While Theorem 1.1(ii) was used in three of the four specific, computations of Sec. 2, its main applications are really more general. Indeed [5, Corollaries 10 and 12] and [6, Proposition 2.5 and Lemma 2.6] assert

Proposition 3.1. *Let G have MJD and let $1 \neq N \triangleleft G$.*

- (i) *If Y is any subgroup of G, then either $Y \supseteq N$ or $YN \triangleleft G$.*
- (ii) *If N is noncyclic, then G/N is a Dedekind group. In particular, if G/N has odd order, then this factor group is abelian.*
- (iii) *Suppose H is a subgroup of G with $H \cap N = 1$. Then H is a Dedekind group and $\mathbb{Q}[H]$ has no nonzero nilpotent elements.*

The fact that H is a Dedekind group in (iii) follows by applying (i) to all subgroups Y of H. In view of the rest of part (iii) above, it is appropriate to record the following from [7, Theorem VI.1.17].

Proposition 3.2. *Let H be a nonabelian group. Then $\mathbb{Q}[H]$ has no nonzero nilpotent elements if and only if $H \cong Q_8 \times E \times A$, where Q_8 is the quaternion group of order 8, E is an elementary abelian 2-group, and A is an abelian group of odd order such that the multiplicative order of 2 modulo $|A|$ is odd.*

Actually, we will only need just one small special case of this result, namely the fact that $\mathbb{Q}[Q_8 \times C_3]$ has nonzero nilpotent elements, and this can be simply verified. Indeed, let $\langle z \rangle = \mathfrak{Z}(Q_8)$ and let $1, a, b, c$ be coset representatives for $\langle z \rangle$ in Q_8. Furthermore, let $C_3 = \langle t \rangle$ with $t^3 = 1$. Then $\alpha = (1 - z)(1 - t)(a + bt + ct^2)$ is a nonzero nilpotent element of $\mathbb{Z}[Q_8 \times C_3]$ since the relations $a^2 = b^2 = c^2 = z$, $ba = abz$, $cb = bcz$, and $ac = caz$ imply that

$$\alpha^2 = (1 - z)^2(1 - t)^2\big(z(1 + t + t^2) + (1 + z)(bc + ab\,t + ca\,t^2)\big) = 0.$$

With this, we can begin our work on groups having order divisible by 6. The following few results are somewhat more general than is needed.

Lemma 3.3. *Suppose G has MJD, let π be a set of primes and let P be a normal nilpotent π-subgroup of G. If there exists a π'-subgroup X of G that acts nontrivially* *on P by conjugation, then P is an elementary abelian p-subgroup of G for some prime $p \in \pi$. Furthermore, if $x \in X$ acts nontrivially on P, then P is an irreducible* $\mathrm{GF}(p)[\langle x\rangle]$*-module.*

Proof. Since the MJD property is inherited by subgroups, we can assume that $G = PX$. Suppose P_0 is a nonidentity normal subgroup of P, with P_0 stable under X. Then $P_0 \triangleleft G$ and, since X does not contain P_0, Proposition 3.1(i) implies that $P_0X \triangleleft G$. Furthermore, since $P \triangleleft G$, we conclude that $[P, X] \subseteq P \cap P_0X = P_0$, so X acts trivially on P/P_0. In particular, if there exist $1 \neq P_1, P_2 \triangleleft P$ with $P_1 \cap P_2 = 1$ and with both stable under X, then $[P, X] \subseteq P_1 \cap P_2 = 1$ and X acts trivially on P, a contradiction. Thus, since P is nilpotent and has characteristic Sylow subgroups, we conclude that P is a p-group.

Next, if the Frattini subgroup $\Phi(P)$ is nontrivial, then by the above with $P_0 = \Phi(P)$, we conclude that X centralizes the Frattini quotient $P/\Phi(P)$. Thus, since X is a p'-group, it follows that X acts trivially on P, again a contradiction. Hence $\Phi(P) = 1$, so P is an elementary abelian p-group and consequently it is a module for $\mathrm{GF}(p)[X]$. Since X is a p'-group, $\mathrm{GF}(p)[X]$ is semisimple, so all its modules are completely reducible.

In particular, if P is not an irreducible $\mathrm{GF}(p)[X]$-module, then $P = P_1 \times P_2$, where P_1 and P_2 are nonidentity $\mathrm{GF}(p)[X]$-submodules. But if this happens, then P_1 and P_2 are X-stable, so we know that X acts trivially, a contradiction. Thus $\mathrm{GF}(p)[X]$ acts irreducibly on P. We can of course apply the latter to any cyclic subgroup $\langle x\rangle$ of X that acts nontrivially on P. □

Next, we need

Lemma 3.4. *Let G have MJD, let $N \triangleleft G$, and let H be a maximal normal subgroup of N. Then $|G : \mathfrak{N}_G(H)| \leq 3$.*

Proof. Say $|G : \mathfrak{N}_G(H)| = n$ and choose coset representatives $g_1, g_2, \ldots, g_n$ for the normalizer subgroup. Then the G-conjugates of H are $H_i = H^{g_i}$, and these are all distinct maximal normal subgroups of N. Let e_N and e_H be the principal idempotents of $\mathbb{Q}[N]$ and $\mathbb{Q}[H]$, respectively. Since $e_He_N = e_Ne_H = e_N$, it follows that $f = e_H - e_N$ is a nonzero idempotent in $\mathbb{Q}[G]$. Indeed, if we set $f_i = f^{g_i} = e_{H_i} - e_N$, then each f_i is also a nonzero idempotent in $\mathbb{Q}[G]$. Now, each H_i is maximal normal in N, so $H_iH_j = N$ for each $i \neq j$, and hence $e_{H_i}e_{H_j} = e_N$. This easily implies that $f_if_j = 0$ and therefore $e = f_1 + f_2 + \cdots + f_n$ is a nonzero idempotent in $\mathbb{Q}[G]$. Indeed, e is central, since conjugation by G permutes the summands and hence fixes e.

Since $e \neq 0$ and $\mathbb{Q}[G]$ is semisimple, there exists an irreducible representation $\theta\colon \mathbb{Q}[G] \to \mathrm{M}_t(D)$ with $\theta(e) \neq 0$. Since $\theta(f_i) = \theta(f^{g_i}) = \theta(f)^{\theta(g_i)}$, it follows that all the $\theta(f_i)$ are conjugate in $\mathrm{M}_t(D)$. In particular, if one $\theta(f_i)$ is zero, then all are zero and $\theta(e) = 0$, a contradiction. Thus $\theta(e)$ is a sum of n nonzero orthogonal idempotents in $\mathrm{M}_t(D)$. This implies that $n \leq t$, and since $t \leq 3$ by Theorem 1.1(i), the result follows. □

Lemma 3.5. *Let G have MJD and let p be the smallest prime dividing $|G|$. Then G has a normal p-complement.*

Proof. If G does not have a normal p-complement, then a theorem of Frobenius implies that G has a nonidentity p-subgroup P and a q-element x that acts nontrivially on P. Here, of course, q is a prime different from p. If $X = \langle x \rangle$, then $H = PX$ is a subgroup of G, and H also satisfies MJD. By Lemma 3.3, P is an elementary abelian p-group and X acts irreducibly on P. If P is cyclic, then q divides $p - 1$, a contradiction since p is the smallest prime dividing $|G|$.

Thus P is not cyclic, and we can choose a maximal subgroup P_1 of P, with $P_1 \neq 1$. Of course P_1 is not X-stable, but $|X : \mathfrak{N}_X(P_1)| \leq |H : \mathfrak{N}_H(P_1)| \leq 3$ by the previous lemma. Since X is a q-group, we conclude that $q = 3$ and $p = 2$. Also $\mathfrak{N}_X(P_1)$ does not act irreducibly on P, so it must act trivially by Lemma 3.3 again. Thus, X acts on P like $X/\mathfrak{N}_X(P_1)$, a group of order 3. Consequently, $P \cong C_2 \times C_2$, a fours group, and $H = PX$ is the type of group considered by Lemma 2.4. But that lemma asserts that H does not have MJD, a contradiction, and therefore G has a normal p-complement. □

We now prove our main result, Theorem 1.3, which we repeat for convenience.

Theorem 3.6. *Let G be a finite nonabelian $2,3$-group with order divisible by 6. Then $\mathbb{Z}[G]$ satisfies MJD if and only if*

(i) $G = \mathrm{Sym}_3$, *the symmetric group of degree* 3,
(ii) $G = \langle x, y \mid x^3 = 1,\ y^4 = 1,\ y^{-1}xy = x^{-1} \rangle$, *a group of order* 12, *or*
(iii) $G = Q_8 \times C_3$, *the direct product of the quaternion group of order* 8 *with the cyclic group of order* 3.

Proof. Suppose $\mathbb{Z}[G]$ satisfies MJD, and let us first assume that G is not nilpotent. By the previous lemma, we see that G has a normal 2-complement P. Thus $P \neq 1$ is the unique Sylow 3-subgroup of G. If Q is a Sylow 2-subgroup of G, then since G is not nilpotent, Q acts nontrivially on P by conjugation. It follows from Lemma 3.3 that P is an elementary abelian 3-group and an irreducible $\mathrm{GF}(3)[\langle y \rangle]$-module for all y in Q that act nontrivially on P. We can of course choose $y \in Q \setminus \mathfrak{C}_Q(P)$ with $y^2 \in \mathfrak{C}_Q(P)$. Thus y acts like an element of order 2, so it follows that P is cyclic of order 3 and y has dihedral action on P. Indeed, since $|\mathrm{Aut}(P)| = 2$, we have $|Q : \mathfrak{C}_Q(P)| = 2$. In particular, $Q = \mathfrak{C}_Q(P)\langle y \rangle$.

Now $N = \mathfrak{C}_Q(P)$ is normalized by both Q and P, so $N \triangleleft G$. If $\langle y \rangle$ does not contain N, then by Proposition 3.1(i), $Q/N = N\langle y \rangle/N \triangleleft G/N$. But this implies that $[P, y] \subseteq P \cap N\langle y \rangle = P \cap Q = 1$ and y centralizes P, a contradiction. Thus $\langle y \rangle \supseteq N$, so $Q = \langle y \rangle$. In particular, if $P = \langle x \rangle$, then G is generated by x and y with

$y^{2^n} = 1$, $x^3 = 1$ and $y^{-1}xy = x^{-1}$. Lemma 2.3 now implies that $n = 1$ or 2, and these are the listed groups of types (i) and (ii).

Next, suppose that $G = Q \times P$ is nilpotent, where Q is a nonidentity Sylow 2-subgroup and where P is a nonidentity Sylow 3-subgroup. By Proposition 3.1(iii), with $N = Q$, we see that P is abelian, and hence Q is nonabelian. Furthermore, by Proposition 3.1(iii), this time with $N = P$, we see that Q is a Dedekind group. Thus $Q = Q_8 \times E$ where E is an elementary abelian 2-group.

Since $P \neq 1$, we can let $C_3 \subseteq P$ be a subgroup of order 3, and we set $H = Q_8 \times C_3 \subseteq G$. Since 2 has even order 2 modulo 3, Proposition 3.2 or the comments after it show that $\mathbb{Q}[H]$ has nonzero nilpotent elements. Thus Proposition 3.1(iii) implies that $H \cap N \neq 1$ for all nonidentity normal subgroups N of G. In particular, since $H \cap E = 1$, we have $E = 1$. Furthermore, P has no nonidentity subgroup disjoint from C_3, so P is cyclic. If $|P| \geq 9$, then P has a cyclic subgroup C_9 of order 9 and then $Q_8 \times C_9$ is a subgroup of G. But all subgroups of G have MJD, while $Q_8 \times C_9$ does not by [5, Lemma 14]. We conclude that $P = C_3$ and hence that G is the group of type (iii).

It remains to observe that the three listed groups have the MJD property. To start with, [3, Example 3.3] shows that $\mathbb{Z}[\mathrm{Sym}_3]$ has the additive and hence multiplicative Jordan decomposition property. In fact, by [3, Theorem 3.4], this holds for all dihedral groups of order $2p$ with p an odd prime. Next, [1, Proposition 3.1] handles the group (ii) of order 12. Indeed, [2, Theorem 6.1] shows that the MJD property holds for all "generalized quaternion groups" of order $4p$, with p any odd prime. Finally, [5, page 115] yields the result for the group $Q_8 \times C_3$ of order 24. □

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