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Filtrations in Semisimple Lie Algebras, III

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Dedicated to S.K. Jain on the occasion of his retirement

Abstract. This is the third in a series of papers. The first two, by Yiftach Barnea and this author, study the maximal bounded $\mathbb{Z}$-filtrations of the finite-dimensional simple Lie algebras over the complex numbers. Those papers obtain a complete characterization for all but the five exceptional Lie algebras, namely the ones of type G_2, F_4, E_6, E_7 and E_8. Here, we fill in the missing step for the algebra G_2. The proof is computational and uses MAGMA, a computer algebra package, to handle the 7×7 matrices that occur.

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1. Preliminaries

Let L be a Lie algebra over the complex field K. A $\mathbb{Z}$-filtration $\mathcal{F} = \{F_i \mid i \in \mathbb{Z}\}$ of L is a collection of K-subspaces

$$\cdots \subseteq F_{-2} \subseteq F_{-1} \subseteq F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

indexed by the integers $\mathbb{Z}$ such that $[F_i, F_j] \subseteq F_{i+j}$ for all $i, j \in \mathbb{Z}$. One usually also assumes that $\bigcup_i F_i = L$ and $\bigcap_i F_i = 0$. In particular, F_0 is a Lie subalgebra of L and each F_i is an F_0-Lie submodule of L. Furthermore, we say that the filtration is bounded if there exist integers ℓ and ℓ' with $F_\ell = 0$ and $F_{\ell'} = L$. In this case, it is clear that each F_i, with $i < 0$, is ad-nilpotent on L.

If A is any finite-dimensional Lie algebra then the Ado-Iwasawa Theorem (see [4, Chapter VI]) implies that A embeds in some $L = \mathfrak{gl}_n(K)$ and therefore we obtain a filtration of L with $F_{-1} = 0$, $F_0 = A$ and $F_1 = L$. Thus, it is clearly hopeless to try to classify all the bounded filtrations of the various $\mathfrak{gl}_n(K)$, even if only up to isomorphism. Nevertheless, there is something that can be done.

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Again, let $\mathcal{F}$ be a filtration of an arbitrary Lie algebra L. If $\mathcal{G} = \{G_i \mid i \in \mathbb{Z}\}$ is a second such filtration, we say that $\mathcal{G}$ contains $\mathcal{F}$, or $\mathcal{G}$ is larger than $\mathcal{F}$, if $G_i \supseteq F_i$ for all i. In particular, it makes sense to speak about maximal bounded filtrations, and the goal of [1, 2], the first two papers in this series, is to classify such filtrations $\mathcal{F}$ when L is a simple Lie algebra over the complex numbers.

This classification is achieved in four key steps. The first step shows that F_0, the 0-component of $\mathcal{F}$, contains a Cartan subalgebra H of L. Since each component F_i is then an $\operatorname{ad} H$-submodule of L, it follows easily that these F_i are sums of certain $\operatorname{ad} H$-eigenspaces, that is root spaces, L_α. Note that it is necessary to allow α to equal 0 here, with $L_0 = H$. The second step makes this statement more precise by proving that $\mathcal{F} = \mathcal{F}_\lambda$ is a dual filtration. Here λ is a functional on the real root space of L, and each F_i is given by the sum of those L_α with $\lambda(\alpha) \leq i$. It turns out that not every dual filtration is maximal, and the third step shows that $\mathcal{F}_\lambda$ is maximal if and only if λ takes on integer values on an $\mathbb{R}$-basis of roots for the root space. Finally, the fourth step precisely determines these maximal λ by better understanding the $\mathbb{R}$-bases that occur.

Paper [2] deals with the fourth step, while [1] essentially handles the first three. Indeed, all that is missing is the verification of the first step in the case of the five exceptional Lie algebras, namely those of type G_2, F_4, E_6, E_7 and E_8. In this paper, we supply the verification for the smallest exception, namely G_2. The method of proof is somewhat computational, using the precise embedding of G_2 in the Lie algebra B_3. Specifically, we have $G_2 \subseteq B_3 \subseteq \mathfrak{gl}_7(K)$, and consequently our argument requires dealing with certain 7×7 matrices. For this, we use MAGMA, a computer algebra package.

We begin with some preliminary observations. For the most part, these are fairly immediate consequences of the results in [5] and [1, Section 2]. It is first necessary to deal with matrix rings. Here, of course, the filtrations satisfy $F_iF_j \subseteq F_{i+j}$, and we allow K to be a division ring. See [5] for basic definitions.

Lemma 1.1. *Let $R = \mathbf{M}_n(K)$ be the ring of $n \times n$ matrices over the division ring K and let $S \cong \oplus \sum_{i=1}^{k} \mathbf{M}_{n_i}(K)$ be the subring of R consisting of all block diagonal matrices of the form* $\operatorname{diag}(s_1, s_2, \ldots, s_k)$*, where $s_i \in \mathbf{M}_{n_i}(K)$. If $\mathcal{F} = \{F_a \mid a \in \mathbb{Z}\}$ is a maximal bounded $\mathbb{Z}$-filtration of S, then there exists a maximal bounded $\mathbb{Z}$-filtration $\mathcal{G} = \{G_a \mid a \in \mathbb{Z}\}$ of R with $G_a \cap S = F_a$ for all $a \in \mathbb{Z}$.*

Proof. By [5, Theorem 3.6], $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_k$, where each $\mathcal{F}_i$ is a maximal bounded filtration of $\mathbf{M}_{n_i}(K)$. Furthermore, by choosing an appropriate basis, we can assume that each $\mathcal{F}_i$ is a weight filtration. In other words, if N_i denotes the set of integers that correspond to the row and column positions of $\mathbf{M}_{n_i}(K)$ in $\mathbf{M}_n(K)$, then $\mathcal{F}_i$ is determined by a weight function $\omega_i\colon N_i \to \mathbb{Z}$. But $N = \{1, 2, \ldots, n\}$ is the disjoint union of the various N_i, so we can define $\omega\colon N \to \mathbb{Z}$ to extend all of the functions ω_i. Finally, the filtration $\mathcal{G} = \{G_a \mid a \in \mathbb{Z}\}$ of R determined by ω is maximal bounded, by [5, Theorem 3.6], and the definition of weight filtration clearly implies that $G_a \cap S = F_a$ for all $a \in \mathbb{Z}$. □

In the above situation, we say that $\mathcal{G}$ covers $\mathcal{F}$. Indeed, we will use this notation in all of the various contexts below. In the remainder of this paper, K will denote an algebraically closed field of characteristic 0, essentially K is the complex numbers, and we consider the finite-dimensional simple Lie algebras over K.

Lemma 1.2. *Let L be a simple Lie algebra over K and assume that $L \subseteq \mathfrak{gl}_n(K) \subseteq \mathbf{M}_n(K) = R$. If $\mathcal{F} = \{F_a \mid a \in \mathbb{Z}\}$ is a maximal bounded $\mathbb{Z}$-filtration of L, then there exists a maximal bounded $\mathbb{Z}$-filtration $\mathcal{G} = \{G_a \mid a \in \mathbb{Z}\}$ of R such that $G_a \cap L = F_a$ for all $a \in \mathbb{Z}$.*

Proof. First assume that L acts irreducibly on the vector space $V = K^n$. Then, following [1, Section 2], we let $\mathcal{F}^R = \{\widetilde{F}_a \mid a \in \mathbb{Z}\}$ to be the family of subspaces of R that are defined by

$$\widetilde{F}_a = \sum F_{i_1} F_{i_2} \cdots F_{i_t}$$

where the sum is over all $t \geq 0$ and all subscripts with $i_1 + i_2 + \cdots + i_t \leq a$. According to [1, Lemma 2.4], $\mathcal{F}^R$ is a bounded $\mathbb{Z}$-filtration of R, and hence we can extend $\mathcal{F}^R$ to $\mathcal{G} = \{G_a \mid a \in \mathbb{Z}\}$, a maximal bounded filtration of R. Since $\mathcal{G}_L = \{G_a \cap L \mid a \in \mathbb{Z}\}$ is a bounded $\mathbb{Z}$-filtration of L containing $\mathcal{F}$, by [1, Lemma 2.1], the maximality of $\mathcal{F}$ now implies that $\mathcal{F} = \mathcal{G}_L$, as required.

For the general case, we use Weyl's Theorem [3, Theorem 6.3], which asserts that L acts completely reducibly on V. Thus, with respect to a suitable basis, $R = \mathbf{M}_n(K)$ contains the subring $S \cong \oplus \sum_{i=1}^{k} \mathbf{M}_{n_i}(K)$ of block diagonal matrices corresponding to the irreducible constituents of this representation of L. In other words, there exist homomorphisms $\phi_i \colon L \to \mathfrak{gl}_{n_i}(K) \subseteq \mathbf{M}_{n_i}(K)$ that are either irreducible representations of L or zero maps, and with at least one ϕ_i not zero. Now if $\phi_i \neq 0$, then $\phi_i(\mathcal{F})$ is a maximal bounded $\mathbb{Z}$-filtration of $\phi_i(L) \cong L$, so by the above, $\phi_i(\mathcal{F})$ is covered by $\mathcal{G}_i$, a maximal bounded filtration of $\mathbf{M}_{n_i}(K)$. On the other hand, if $\phi_i = 0$, then $\phi_i(L) = 0$, so $\phi_i(\mathcal{F})$ is obviously covered by any maximal bounded filtration $\mathcal{G}_i$ of $\mathbf{M}_{n_i}(K)$. Since F_a is a subdirect product of its images $\phi_i(F_a)$, it follows from [5, Theorem 3.6] that $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_k$ is a maximal bounded filtration of S with $\mathcal{G}_L \supseteq \mathcal{F}$. Finally, we can apply the preceding lemma to find a maximal bounded filtration $\mathcal{H} = \{H_a \mid a \in \mathbb{Z}\}$ of R that covers $\mathcal{G}$. Then $\mathcal{H}_L \supseteq \mathcal{F}$, and the maximality of $\mathcal{F}$ yields the result. □

This has two consequences of interest. First, we have

Lemma 1.3. *Let $\mathcal{F} = \{F_a \mid a \in \mathbb{Z}\}$ be a maximal bounded $\mathbb{Z}$-filtration of the simple K-Lie algebra L. If $x \in F_0$ and if $x = x_s + x_n$ is its Jordan decomposition in L, then the semisimple part x_s and the nilpotent part x_n both belong to F_0.*

Proof. Using an irreducible representation of L, we embed L in the Lie algebra $\mathfrak{gl}_n(K) \subseteq \mathbf{M}_n(K) = R$. Therefore, by the previous lemma, $\mathcal{F}$ is covered by a maximal filtration $\mathcal{G} = \{G_a \mid a \in \mathbb{Z}\}$ of R and, in particular, $F_0 = G_0 \cap L$ and $x \in F_0 \subseteq G_0$. Now, by [3, Theorem 6.4], $x = x_s + x_n$ is also the usual Jordan decomposition of x in the matrix ring R. Thus, by [3, Proposition 4.2], $x_s = p(x)$ and $x_n = q(x)$, where p and q are polynomials over K without constant terms.

Since G_0 is a subalgebra of R, it now follows that $x_s, x_n \in G_0$ and consequently $x_s, x_n \in G_0 \cap L = F_0$. □

Furthermore, we have

Lemma 1.4. *Let $L \subseteq \bar{L}$ be simple Lie algebras over K. If $\mathcal{F} = \{F_a \mid a \in \mathbb{Z}\}$ is a maximal bounded $\mathbb{Z}$-filtration of L, then there exists a maximal bounded $\mathbb{Z}$-filtration $\mathcal{G} = \{G_a \mid a \in \mathbb{Z}\}$ of $\bar{L}$ with $F_a = G_a \cap L$ for all $a \in \mathbb{Z}$.*

Proof. Using an irreducible representation of $\bar{L}$, we embed $\bar{L}$ in $\mathfrak{gl}_n(K) \subseteq \mathbf{M}_n(K) = R$. Then $L \subseteq \mathfrak{gl}_n(K)$, so Lemma 1.2 implies that there exists a $\mathbb{Z}$-filtration $\mathcal{H} = \{H_a \mid a \in \mathbb{Z}\}$ of R with $H_a \cap L = F_a$ for all $a \in \mathbb{Z}$. Furthermore, by [1, Lemma 2.1], $\mathcal{H}_{\bar{L}} = \{H_a \cap \bar{L} \mid a \in \mathbb{Z}\}$ is a bounded filtration of $\bar{L}$, and this extends to a maximal bounded filtration $\mathcal{G} = \{G_a \mid a \in \mathbb{Z}\}$ of $\bar{L}$. Note that

$$G_a \cap L \supseteq (H_a \cap \bar{L}) \cap L = H_a \cap L = F_a,$$

so $\{G_a \cap L \mid a \in \mathbb{Z}\}$ is a bounded filtration of L containing $\mathcal{F}$. The maximality of $\mathcal{F}$ now implies that $G_a \cap L = F_a$ for all $a \in \mathbb{Z}$. □

The following is implicit in the work of [1].

Lemma 1.5. *Let $\mathcal{F} = \{F_a \mid a \in \mathbb{Z}\}$ be a maximal bounded filtration of the simple K-Lie algebra of classical type. Then $F_0 \supseteq B$, a Borel subalgebra of L.*

Proof. By [1, Section 5], F_0 contains a Cartan subalgebra H of L, and indeed $\mathcal{F} = \mathcal{F}_\lambda$ for some suitable linear functional λ on the root space. It follows from the definition of $\mathcal{F}_\lambda$ that if α is a root, then at least one of the root spaces L_α or $L_{-\alpha}$ is contained in F_0. With this, it is easy to see that if B/F_{-1} is a Borel subalgebra of F_0/F_{-1} containing $(H + F_{-1})/F_{-1}$, then $B \subseteq F_0$ is a Borel subalgebra of L. □

If L is a K-Lie algebra and if S is a solvable subalgebra, then $S \subseteq B$ where B is a Borel subalgebra of L. If B is uniquely determined by S, then we say that S is uniquely extendible in L. It is clear that if $S \subseteq T \subseteq L$, with S and T both solvable, and if S is uniquely extendible, then so is T. Our reason for introducing this concept is the simple result given below that can be used in concert with the previous two lemmas.

Lemma 1.6. *Let $L \subseteq \bar{L}$ be Lie algebras over K, and let $\bar{B}$ be a Borel subalgebra of $\bar{L}$. If $\bar{B} \cap L$ is uniquely extendible in $\bar{L}$, then $\bar{B} \cap L$ is a Borel subalgebra of L.*

Proof. Obviously, $S = \bar{B} \cap L$ is a solvable subalgebra of both L and $\bar{L}$, and hence $S \subseteq B$, where B is a suitable Borel subalgebra of L. Furthermore, B extends to $\bar{B}_1$, a Borel subalgebra of $\bar{L}$. In other words, we have $S \subseteq \bar{B}$ and $S \subseteq \bar{B}_1$ so, since S is uniquely extendible in $\bar{L}$, we conclude that $\bar{B} = \bar{B}_1 \supseteq B$ and hence $S = \bar{B}_1 \cap L \supseteq B$. Thus $S = B$, as required. □

We close this section with two fairly standard results from Lie theory. We include brief proofs of each for the convenience of the reader.

To start with, we say that the subalgebra S of L is ad-nilpotent if $\operatorname{ad} S$ is nilpotent in its action on L. Certainly this implies that S is a nilpotent and hence solvable subalgebra of L, so $S \subseteq B$ for some Borel subalgebra B of L. Indeed, if L is simple, then S embeds in N, the nilradical of B. The following lemma contains a sufficient condition for such a subalgebra S to be uniquely extendible. Note that the expressions S^k and N^k below are the associative powers of S and N in the endomorphism ring of $V = K^n$, the space of n-tuples over K.

Lemma 1.7. *Let S be a Lie subalgebra of $\mathfrak{gl}_n(K)$ so that S acts on the right on the vector space $V = K^n$. If $VS^n = 0$ but $VS^{n-1} \neq 0$, then S is contained in a unique Borel subalgebra of $\mathfrak{gl}_n(K)$ and hence in a unique Borel subalgebra of any intermediate Lie algebra.*

Proof. Since $VS^n = 0$, S is nilpotent in its action on V and hence ad-nilpotent in its action on $\mathfrak{gl}_n(K)$. If $B = N + H$ is a Borel subalgebra of $\mathfrak{gl}_n(K)$ containing S, then the nilradical N contains S. It follows that $VN^i \supseteq VS^i$ for all i, and we know that $VN^n = 0$. Thus, since $VS^{n-1} \neq 0$, it is clear that $VN^i = VS^i$ for all i. But B is the set of elements of $\mathfrak{gl}_n(K)$ that stabilize the flag $V = VN^0 \supseteq VN^1 \supseteq \cdots \supseteq VN^n = 0$, so since $VN^i = VS^i$ we see that B is uniquely determined by S. □

Finally, we have

Lemma 1.8. *Let S be a solvable Lie subalgebra of $\mathfrak{gl}_n(K)$ closed under Jordan decomposition in its action on $V = K^n$. Then S is the direct sum $S = M + C$, where M is the Lie ideal consisting of nilpotent elements of L and where C is a complementary commutative space of semisimple elements.*

Proof. S is contained in a Borel subalgebra $B = N + H$ of $\mathfrak{gl}_n(K)$, where N is the Lie ideal of all nilpotent elements of B and where H is a Cartan subalgebra, a commutative semisimple complement. It follows that $M = N \cap S$ is the subspace of S consisting of all nilpotent elements of S. Furthermore, M is a Lie ideal of S with S/M abelian. The goal is to find a semisimple complementary subspace for M in S, and we proceed by induction on $\dim_K S$.

Suppose first that S has a semisimple element x not contained in its center. Then $\operatorname{ad} x$ is semisimple in its action on S, so S is the direct sum $S = S_0 + S_1$ where $S_0 = \mathfrak{C}_S(x)$ and S_1 is an $\operatorname{ad} x$-stable complement. Clearly S_0 is a Lie subalgebra of S and $\dim S_0 < \dim S$ since x is not central in S. Furthermore, if $y \in S_0$, then the nilpotent and semisimple parts of y are polynomials in y and hence also commute with x. In other words, S_0 is closed under Jordan decomposition, so by induction $S_0 = M_0 + C_0$. On the other hand, S_1 is spanned by eigenvectors of $\operatorname{ad} x$ with nonzero eigenvalues and hence each such eigenvector is contained in $[S, S] \subseteq M$. Thus $S = M + C_0$, and C_0 is the required complement of semisimple elements.

It now suffices to assume that all semisimple elements of S are central in S, and we let C denote the set of all such elements. We show that C is a subspace of S. To this end, let $x, y \in C$. Then x and y commute, so they are commuting diagonalizable elements and hence they can be simultaneously diagonalized. Thus $Kx + Ky$ consists of semisimple elements and hence is contained in C. It follows that C is a subspace and since $M + C$ contains the nilpotent and semisimple parts of all elements of S, we conclude that $S = M + C$. □

2. The Lie algebra G_2

As we mentioned, in order to complete the classification of the maximal bounded $\mathbb{Z}$-filtrations of the simple Lie algebras, we must show, in the case of the exceptional Lie algebras, that the 0-component of such filtrations contains a Cartan subalgebra. The goal of this section is to prove this for G_2, and indeed we have

Theorem 2.1. *Let $\mathcal{F} = \{F_i \mid i \in \mathbb{Z}\}$ be a maximal bounded $\mathbb{Z}$-filtration of the Lie algebra L of type G_2 over the algebraically closed field K of characteristic 0. Then F_0 contains a Cartan subalgebra of L.*

Proof. We use the precise description of G_2 as given in [3, Section 19.3]. Indeed, those few pages describe a faithful 7-dimensional representation of the Lie algebra and show that $L \subseteq \overline{L}$ where $\overline{L}$ is of type B_3. Our argument requires some matrix and vector space computations and, for this, we use MAGMA, a computer algebra package. The original version of this manuscript, containing an annotated write up of the fairly simple code we require, can be found on the author's web page

`www.math.wisc.edu/~passman/abstracts.html`.

A complete MAGMA input and output text file is also available there.

Now let $\mathcal{F} = \{F_i \mid i \in \mathbb{Z}\}$ be a maximal bounded $\mathbb{Z}$-filtration of the Lie algebra L. Then it follows from Lemma 1.4 that $\overline{L}$ has a maximal bounded $\mathbb{Z}$-filtration $\mathcal{G} = \{G_i \mid i \in \mathbb{Z}\}$ such that $F_i = G_i \cap L$ for all $i \in \mathbb{Z}$. Furthermore, by Lemma 1.5, G_0 contains a Borel subalgebra $\overline{B}$ of $\overline{L}$, and hence $F_0 = G_0 \cap L \supseteq \overline{B} \cap L$, a solvable subalgebra of L. In particular, $\overline{B} \cap L \subseteq B$, a Borel subalgebra of L. Since all Borel subalgebras of L are conjugate, we can assume that B is as described in [3, Section 19.3]. Furthermore, let $\overline{N}$ denote the nilradical of $\overline{B}$, and let N be the nilradical of B. From [3, Sections 1.2 and 19.3], we have

$$\dim N = 6, \qquad \dim B = 6 + 2 = 8, \qquad \dim L = 8 + 6 = 14$$
$$\dim \overline{N} = 9, \qquad \dim \overline{B} = 9 + 3 = 12, \qquad \dim \overline{L} = 12 + 9 = 21.$$

Our computations use the basis $\{a, b, c, d, e, f\}$ for N as described in [3]. These basis members are, in fact, all root vectors corresponding, respectively, to the roots α, β, $\alpha+\beta$, $2\alpha+\beta$, $3\alpha+\beta$, and $3\alpha+2\beta$ of G_2, where α and β are simple.

We note that the 7×7 matrices for c and d are given by

$$c = \begin{bmatrix} 0 & 0 & 0 & 0 & t & 0 & 0 \\ -t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$d = \begin{bmatrix} 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -t & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $t = \sqrt{2}$. Furthermore, the nonzero Lie products of the basis elements are easily found to be

$$[a,b] = -c, \quad [a,c] = -2d, \quad [a,d] = -3e, \quad [b,e] = f, \quad [c,d] = -3f,$$

and hence we have

Lemma 2.2. *The terms of the lower central series of N are given by*

$$\begin{aligned} N^{[1]} &= Ka + Kb + Kc + Kd + Ke + Kf \\ N^{[2]} &= Kc + Kd + Ke + Kf \\ N^{[3]} &= Kd + Ke + Kf \\ N^{[4]} &= Ke + Kf \\ N^{[5]} &= Kf. \end{aligned}$$

Furthermore, N is contained in a unique Borel subalgebra of the algebra $\overline{L}$. In particular, if $N \subseteq L \cap \overline{B}$, then $L \cap \overline{B}$ is a Borel subalgebra of L.

Proof. The terms of the lower central series are trivial to compute from the above commutator relations. For the last part, we want to show that $S = N$ is uniquely extendible in $\overline{L}$. For this, we first observe that N is ad-nilpotent on $\overline{L}$. Furthermore, $\overline{L}$ admits the same 7-dimensional module V as does L. Thus, in view of Lemma 1.7, it suffices to show that $N^6 \neq 0$ in its action on V. But $f \in N^{[5]} \subseteq N^5$ and we easily check that the matrix product fa is not 0. Thus $N^6 \neq 0$, and consequently Lemma 1.6 yields the result. □

It can be shown that N contains an element having one 7×7 Jordan block in its action on V, and such regular nilpotent elements are known to be contained in unique Borel subalgebras. Next, we note that

$$\dim \overline{L} \geq \dim(L + \overline{B}) = \dim L + \dim \overline{B} - \dim(L \cap \overline{B})$$

and hence

$$\dim(L \cap \overline{B}) \geq \dim L + \dim \overline{B} - \dim \overline{L} = 14 + 12 - 21 = 5.$$

Since $L \cap \overline{B} \subseteq B$, it also follows that $\dim(L \cap \overline{B}) \leq 8$.

Now $L \cap \overline{B} = F_0 \cap \overline{B}$ and, since $\overline{B}$ is closed under taking semisimple and nilpotent parts, the same is true of $L \cap \overline{B}$ by Lemma 1.3. Furthermore, recall from [3, Theorem 6.4] that the Jordan decomposition of any element of L in its action on V and in its ad-action on L are identical. It therefore follows from Lemma 1.8 that $L \cap \overline{B} = M + C$ where $M = L \cap \overline{N} \subseteq N$ and where C is a semisimple complement. If $\dim C \geq 2$, then $\dim C = 2$ and C is a Cartan subalgebra of L contained in F_0. Thus, we can assume that either $C = 0$ or $C = Kh$ has dimension 1. Indeed, by taking a suitable conjugate if necessary, we can assume that $h \in H \subseteq B$ where H is any Cartan subalgebra of our choosing.

Note that M is properly smaller than N since, if $N \subseteq L \cap \overline{B}$, then Lemma 2.2 implies that $L \cap \overline{B} = B$ contains a Cartan subalgebra of L. Thus $\dim(L \cap \overline{B}) < \dim N + 1 = 7$, and hence there are just two cases remaining to be considered. In case 1, we have $\dim(L \cap \overline{B}) = 5$ and either $L \cap \overline{B} = M$ or $L \cap \overline{B} = M + Kh$, where $M = N \cap \overline{B}$ and where h is some nonzero element of H. On the other hand, in case 2, $\dim(L \cap \overline{B}) = 6$ and, since $L \cap \overline{B} \neq N$, we have $L \cap \overline{B} = M + Kh$, where M and h are as above. Furthermore, we can assume that H is the Cartan subalgebra which we now describe.

Following [3, Section 19.3], we note that a Cartan subalgebra of $\overline{L}$ is diagonal with basis $d_1 = e_{22} - e_{55}$, $d_2 = e_{33} - e_{66}$ and $d_3 = e_{44} - e_{77}$, where of course $\{e_{ij}\}$ is the set of matrix units in $\mathbf{M}_7(K)$. Furthermore, a Cartan subalgebra $H \subseteq B$ of L is given by all elements of the form $h = k_1 d_1 + k_2 d_2 + k_3 d_3$ with $k_1, k_2, k_3 \in K$ and $k_1 + k_2 + k_3 = 0$. In particular, if all k_i are nonzero, then $\operatorname{rank} h = 6$. Thus, up to scalar factors, there are just three nonzero members of H of rank less than 6, and these all have rank 4. Specifically, we take these three elements to be $h_1 = d_1 - d_2$, $h_2 = d_2 - d_3$ and $h_3 = d_3 - d_1$. For convenience, let us define

$$N_a = Ka + Kc + Kd + Ke + Kf = Ka + [N, N] \subseteq N,$$

and

$$N_b = Kb + Kc + Kd + Ke + Kf = Kb + [N, N] \subseteq N.$$

Then, we have

Lemma 2.3. *Let H be the Cartan subalgebra of L contained in the diagonal subspace of $\mathbf{M}_7(K)$. Then $H \subseteq B$ and, up to a scalar multiple, there are just three nonzero elements of H having rank less than 6. These elements, h_1, h_2 and h_3, all have rank 4 and satisfy $\alpha(h_1) = -1$, $\beta(h_1) = 2$, $\alpha(h_2) = 1$, $\beta(h_2) = -1$ and $\alpha(h_3) = 0$, $\beta(h_3) = -1$. Furthermore, suppose M is a Lie subalgebra of N of codimension 1. Then $M \triangleleft N$, $M \supseteq [N, N]$ and, if M is $\operatorname{ad} h$-stable with $h = h_1$, h_2 or h_3, then $M = N_a$ or N_b.*

Proof. The values of $\alpha(h_i)$ and $\beta(h_i)$ are easily computed from the formulas $[h_i, a] = \alpha(h_i)a$ and $[h_i, b] = \beta(h_i)b$. It remains to consider the Lie subalgebra M of codimension 1 in N. Since normalizers grow in nilpotent algebras, it follows that $M \triangleleft N$ and of course N/M is abelian. Thus $M \supseteq [N, N]$, and note that $N = Ka+Kb+[N, N]$. Finally, suppose $h = h_1$, h_2 or h_3 and that M is $\operatorname{ad} h$-stable. Since $\alpha(h) \neq \beta(h)$, h has distinct eigenvalues on $Ka + Kb$, with eigenvectors a and b. Thus we conclude that the only possibilities for M are $Ka + [N, N] = N_a$ or $Kb + [N, N] = N_b$. □

It follows from the values of $\alpha(h)$ and $\beta(h)$ given above that h_1, h_2 and h_3 are not regular elements of H. Since $\alpha(h_3) = 0$, h_3 is, in some sense, the worst offender.

Now, as is well known, $L = G_2$ has no nonzero representation of degree less than 7, and it has a unique irreducible representation of degree equal to 7. Indeed, this is a consequence of Weyl's dimension formula and the fact that irreducible representations are uniquely determined by their highest weight. An explicit formula for the degrees of the irreducible representations of G_2 can be found in [3, page 140]. The unique representation of degree 7 is obviously the representation described in [3, Section 19.3], where L acts on the right on a 7-dimensional vector space V. Furthermore, since $\dim \overline{L}/L = 21 - 14 = 7$, we see that the adjoint representation of L on $\overline{L}$ has the factor module $\overline{L}/L \cong V$. In other words, we can compute certain invariants for the adjoint action of L on $\overline{L}/L$ by considering the matrix action of L on V. For convenience, let $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be the natural basis for V corresponding to the matrix representation of L. We can now handle the two cases in turn. We start with

Lemma 2.4. *Case* 1 *cannot occur.*

Proof. Suppose, by way of contradiction, that $\dim(L \cap \overline{B}) = 5$. Then

$$\dim(L + \overline{B}) = \dim L + \dim \overline{B} - \dim(L \cap \overline{B}) = 14 + 12 - 5 = 21,$$

and hence $L + \overline{B} = \overline{L}$. Furthermore, since $\overline{B}$ and $L \cap \overline{B}$ are $\operatorname{ad}(L \cap \overline{B})$-submodules of $\overline{L}$, we conclude that

$$V \cong \frac{\overline{L}}{L} = \frac{L + \overline{B}}{L} \cong \frac{\overline{B}}{L \cap \overline{B}}$$

as $(L \cap \overline{B})$-modules.

If $M = L \cap \overline{B} \subseteq N$, then M has codimension 1 in N, and hence $M \supseteq [N, N]$ by Lemma 2.3. Note also that $M \subseteq N \subseteq \overline{N}$ and that $\overline{N} \triangleleft \overline{B}$ with $\overline{B}/\overline{N}$ being abelian of dimension 3. It follows that $\overline{N}/M$ is an M-submodule of $\overline{B}/M$ of dimension 4 and that M acts trivially on the quotient $\overline{B}/\overline{N}$. Translating this to the module V, we conclude that $VM \supseteq V[N, N]$ has dimension at most 4. But our MAGMA computations show that $\dim V[N, N] = 5$, so this possibility cannot occur.

It remains to assume that $L \cap \overline{B} = M + Kh$, where $M = N \cap \overline{B}$ and h is a nonzero element of H. Obviously, $\dim M = 4$ here, so M has codimension 2 in N. Since $M \subseteq N \subseteq \overline{N}$, in this situation we have $L \cap \overline{B} = M + Kh \subseteq \overline{N} + Kh \subseteq \overline{B}$

and $\overline{N}+Kh$ is an ideal of $\overline{B}$ of codimension 2. Hence $L\cap\overline{B}$ acts trivially on this quotient. Translating to the module V, we conclude that $\dim V(L\cap\overline{B})\leq 5$. In particular, $\operatorname{rank} h\leq 5$ and, as we have indicated, this implies that $h=h_1,h_2$, or h_3 up to a scalar factor. Since normalizers grow in N, we have $M\lhd M_1\lhd N$ with $\dim M_1=5$. Thus $M_1\supseteq[N,N]\supseteq Kc+Kd$ and hence, since M has codimension 1 in M_1, we have $M\cap(Kc+Kd)\neq 0$. Choose $0\neq m=xc+yd\in M\cap(Kc+Kd)$ with $x,y\in K$ and not both 0.

We now compute the dimension of $Vm+Vh_i$ for all $i=1,2,3$. To this end, note that $\operatorname{rank} h_i=4$ and indeed Vh_i has a basis consisting of those v_j that correspond to the four columns where the matrix h_i has nonzero diagonal entries. Thus, to compute the dimension of $(Vm+Vh_i)/Vh_i$, we merely form the matrix of m, delete the columns of that matrix corresponding to the basis elements of Vh_i, and determine the rank of the remaining 7×3 matrix. Indeed, to compute this rank, we can certainly delete any zero row or column. When we do this, the matrices we obtain for m, corresponding to h_1, h_2 and h_3, respectively, are

$$\begin{bmatrix} 0 & yt \\ -xt & 0 \\ 0 & -x \\ -yt & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & xt \\ -xt & 0 \\ 0 & y \\ -yt & 0 \end{bmatrix}, \qquad \begin{bmatrix} -xt & 0 & -y \\ -yt & x & 0 \end{bmatrix}.$$

Note that the rank of each of these matrices is equal to 2 provided that one of x or y is nonzero. Thus $\dim(Vm+Vh_i)/Vh_i=2$, so $\dim(Vm+Vh_i)=2+\dim Vh_i=6$, and this is the required contradiction since $\dim V(L\cap\overline{B})\leq 5$. $\square$

Finally, we show that the second case cannot occur. This is surprisingly a bit more complicated. Since $L\cap\overline{B}$ has dimension 6, there are actually just a few possibilities for this subalgebra. But in this case, we know less about its action on the module V.

Lemma 2.5. *Case* 2 *cannot occur.*

Proof. Suppose, by way of contradiction, that $\dim(L\cap\overline{B})=6$. Then

$$\dim(L+\overline{B})=\dim L+\dim\overline{B}-\dim(L\cap\overline{B})=14+12-6=20,$$

and thus $L+\overline{B}$ has codimension 1 in $\overline{L}$. Furthermore, we know that

$$V\cong\frac{\overline{L}}{L}>\frac{L+\overline{B}}{L}\cong\frac{\overline{B}}{L\cap\overline{B}}$$

as $(L\cap\overline{B})$-modules. Under this isomorphism, the submodule $(L+\overline{B})/L$ corresponds to a subspace W of codimension 1 in V. Unfortunately, we will not always have a precise description of this subspace.

We are also given that $L\cap\overline{B}=M+Kh$, where $M=N\cap\overline{B}$ and where h is a nonzero element of H. Certainly, $\dim M=5$, so that M has codimension 1 in N. Since $L\cap\overline{B}=M+Kh\subseteq\overline{N}+Kh$ and $\overline{N}+Kh$ is an ideal of $\overline{B}$, it follows that $L\cap\overline{B}$ acts trivially on the 2-dimensional quotient $\overline{B}/(\overline{N}+Kh)$. Translating this into V, it follows that $\dim W(M+Kh)\leq 6-2=4$. In particular, h has rank at most 4 on W, and hence h has rank at most 5 on V. We conclude, as before, that $h = h_1$, h_2 or h_3.

In addition, note that M is an $\operatorname{ad} h$-stable subalgebra of N of codimension 1, so Lemma 2.3 implies that $M = N_a = Ka + [N,N]$ or $N_b = Kb + [N,N]$. The first possibility is easy to handle. Indeed, since either choice for M is nilpotent, M must act trivially on the 1-dimensional module V/W, and hence $W \supseteq VM$. In particular, if $M = N_a$, then $W \supseteq VN_a$, and the latter subspace has dimension 6 by our computations. Thus $W = VN_a$ and $W(L \cap \overline{B}) = W(M + Kh) = WN_a + Wh_i$ for some $i = 1, 2$ or 3. However, the latter three subspaces have dimension 6, 5 and 5, respectively, and this contradicts the fact that $\dim W(M + Kh) \leq 4$.

It follows that $M \neq N_a$, and hence $M = N_b$. In this case, $W \supseteq VN_b$ and VN_b has dimension 5 with basis $\{v_1, v_3, v_4, v_5, v_6\}$. Thus $W = VN_b + Ku$, where u is a nonzero vector in $U = Kv_2 + Kv_7$. Note that each of h_1, h_2 and h_3 acts on U with eigenvectors v_2 and v_7. First, let us assume that $h = h_1$ or h_2. Then the eigenvalues of h are 1 and 0, with $v_2h_1 = v_2$, $v_7h_1 = 0$, and $v_2h_2 = 0$, $v_7h_2 = v_7$. Since VN_b is h-stable and W is h-stable, it follows that $W \cap U$ must be h-stable. In particular, we must have $W = W_1 = VN_b + Kv_2 = VN_b + Vh_1$ or $W = W_2 = VN_b + Kv_7 = VN_b + Vh_2$. In this case, computations show that $\dim W_1N_b = 5$ and $\dim W_2N_b = 5$, again contradicting the fact that $\dim W(M + Kh) \leq 4$.

It remains to assume that $M = N_b$ and $h = h_3$. Here we have $W = VN_b + Ku$, where $u = xv_2 + yv_7$ with $x, y \in K$ and not both 0. We first obtain a lower bound for the dimension of WN_b. To this end, note that $V(N_b)^2$ has dimension 2 with basis $\{v_4, v_5\}$. Furthermore, $c, d \in N_b$, so $Ku(N_b) \supseteq Kuc + Kud$. We compute $\dim(V(N_b)^2 + Kuc + Kud)/V(N_b)^2)$ by constructing a 2×7 matrix with first row equal to uc and second row equal to ud. Next we delete the fourth and fifth columns, since they correspond to the vectors v_4 and v_5 in $V(N_b)^2$, and then we find the rank of the remaining matrix. Indeed, we can also delete any zero column, and when we do so, we obtain

$$\begin{bmatrix} -xt & y & 0 \\ -yt & 0 & -x \end{bmatrix},$$

a matrix quite similar to the h_3 matrix of the previous lemma. Clearly, this matrix has rank 2 provided x or y is nonzero.

In other words, we have shown that $\dim WN_b/V(N_b)^2 \geq 2$ and hence that $\dim WN_b \geq 4$. Of course, $WN_b \subseteq VN_b$. On the other hand, since $v_2h_3 = -v_2$ and $v_7h_3 = -v_7$, we see that $u = u(-h_3) \in W(L \cap \overline{B})$. Thus $W(L \cap \overline{B})$ contains the direct sum $WN_b + Ku$ and hence $\dim W(L \cap \overline{B}) \geq 5$, again a contradiction. □

The proofs of Lemmas 2.4 and 2.5 appear, in some sense, to be dual to each other. Furthermore, once these two cases are eliminated, we know that only the earlier configurations can occur, and consequently F_0 must contain a Cartan subalgebra of L. This completes the proof of Theorem 2.1. □

Since the subspaces of V considered above all have natural bases, it is clear that the above computations can all be achieved without using a computer algebra package like MAGMA. On the other hand, it is also clear that without such a package, the experimentation required to find such a proof would have made reaching this goal somewhat problematical. Finally, the author would like to thank the referee for his comment on regular nilpotent elements and for suggesting the validity of Lemma 1.8, which made some later arguments considerably cleaner.

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