PRIME LIE RINGS OF DERIVATIONS OF COMMUTATIVE RINGS IN CHARACTERISTIC 2

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Abstract. Let R be a commutative associative ring with 1 and let $\operatorname{Der}(R)$ be the Lie ring of all derivations of R. Suppose that D is a Lie subring and R-submodule of $\operatorname{Der}(R)$. When R is D-prime, we give necessary and sufficient conditions for D to be Lie prime. Since results of this nature are already known for rings R of characteristic different from 2, what is really new here is the characteristic 2 case.

1. Introduction

Throughout, R is always a commutative ring with 1 and $\operatorname{Der}(R)$ is the set of all derivations of R.

For any subset D of $\operatorname{Der}(R)$, a subset V of R is called D-stable if $\delta(V) \subseteq V$ for all $\delta \in D$. An ideal I of R is called a D-ideal if I is D-stable. A ring R is called D-simple if there are no D-ideals of R other than 0 and R. A ring R is called D-prime if for any nonzero D-ideals I, J of R, we have $IJ \neq 0$.

For Lie rings, we have analogous definitions. A Lie ring D is called (Lie) simple if there are no ideals of D other than 0 and D. A Lie ring D is called (Lie) prime if for any nonzero Lie ideals L, M of D, we have $[L, M] \neq 0$.

There is a natural Lie structure on $\operatorname{Der}(R)$ given by $[\delta, \gamma] = \delta\gamma - \gamma\delta$ for any $\delta, \gamma \in \operatorname{Der}(R)$. $\operatorname{Der}(R)$ also has a left R-module structure given by $(r\delta)(x) = r\delta(x)$ for any $r, x \in R$ and $\delta \in \operatorname{Der}(R)$. Note that for any $\delta, \gamma \in \operatorname{Der}(R)$ and $x \in R$, the composition $\delta(x\gamma)$ is given by

$$\delta(x\gamma) = \delta(x)\gamma + x\delta\gamma$$

and we see that the Lie structure and the module structure interact via

$$[x\delta, y\gamma] = xy[\delta, \gamma] + x\delta(y)\gamma - y\gamma(x)\delta$$

for any $x, y \in R$ and $\delta, \gamma \in \operatorname{Der}(R)$.

From now on, we fix the notation that D is a nonzero Lie subring and also an R-submodule of $\operatorname{Der}(R)$.

It is natural to expect that there are nice relations between D-ideals of R and Lie ideals of D. Specifically, if R is D-simple, one may try to show D is simple and if R is D-prime, then we hope that D is prime.

For the simplicity of D, there are many results in the literature. Most of them exclude the characteristic 2 case. However, [Pas98] studies simplicity

Date: Nov 21, 2005.
2000 *Mathematics Subject Classification.* 16W25, 17B40.

of some tensor products and the case of characteristic 2 is included. [Jor00] goes one step further to prove the following very general theorem.

Theorem 1.1 (Jordan)**.** *Suppose that R is D-simple.*

(1) *D is Lie simple except possibly when* $\operatorname{char} R = 2$ *and D is cyclic as an R-module.*
(2) *If* $\operatorname{char} R = 2$ *and $D = R\delta$ is cyclic as an R-module, then D is Lie simple if and only if $\delta(R) = R$.*

For the primeness of D, [JJ78] gave some results under various conditions. Recently, Chebotar and P.-H. Lee [CL] proved that if $D = R\delta$ and $\operatorname{char} R \neq 2$, then the D-primeness of R implies the Lie primeness of D. Moreover, P.-H. Lee and C.-K. Liu have announced the following theorem.

Theorem 1.2 (P.-H. Lee and C.-K. Liu)**.** *Suppose that R is D-prime and* $\operatorname{char} R \neq 2$. *Then D is Lie prime.*

The goal of this paper is to extend this theorem to include the characteristic 2 case and give a complete characterization of the primeness of D.

The structure of the paper is arranged as follows. In section 2, we record some basic facts and then prove the key proposition. We begin section 3, by studying the special case when $\operatorname{char} R = 2$ and D is cyclic as an R-module. This yields the following theorem which is an analog of Theorem 1.1 (2).

Theorem 1.3. *Suppose that R is D-prime. If* $\operatorname{char} R = 2$ *and $D = R\delta \neq 0$, then D is Lie prime if and only if $\delta(\delta(R)\delta(R)) \neq 0$.*

In view of Theorem 1.2 and 1.3, it is natural to expect that there might be a prime analog of Theorem 1.1. More precisely, one may try to show that if R is D-prime and D is not cyclic as an R-module, then D is prime. This turns out to be false. Indeed, we offer examples in Section 5 to show that the primeness of D is not controlled by the number of R-generators of D. One way to overcome this difficulty is via localization. To this end, let $C = R^D$ be the ring of D-constants of R. Then the nonzero elements C^* of C are regular in R and we use $K = C^{-1}C$ to denote the field of fractions of C and $S = C^{-1}R$ to denote the localization of R at C^*. Since each derivation δ of R in D extends uniquely to a derivation $\hat{\delta}$ of S, we let $\hat{D} = \{\, \hat{\delta} \mid \delta \in D \,\}$ and write $\bar{D} = K\hat{D}$. The following theorem is proved in Section 3.

Theorem 1.4. *Suppose that R is D-prime. Then*

(1) *D is Lie prime except possibly when* $\operatorname{char} R = 2$ *and $\bar{D}$ is cyclic as an S-module.*
(2) *If* $\operatorname{char} R = 2$ *and $\bar{D} = S\theta$ is cyclic as an S-module, then D is Lie prime if and only if $\theta(\theta(S)\theta(S)) \neq 0$.*
(3) *D is Lie prime if and only if* $\operatorname{char} R \neq 2$ *or* $\dim_K S \neq 2$.
(4) *If D is not Lie prime, then $\bar{D}$ is the unique nonabelian Lie algebra of K-dimension* 2.

Part (1) and (2) are of course a prime analog of Theorem 1.1. Part (3) is perhaps a better characterization of the primeness of D which uses conditions on R instead of conditions on D.

Finally, one may wish to avoid localizations and have conditions stated entirely within R. This is done in Section 4.

The first author would like to thank Professor P.-H. Lee for his informative talk on materials in [CL], thank C.-K. Liu for his bringing [Jor00] to the author's attention, and thank J.-C. Liu for helpful discussions.

2. Key Proposition

The goal of this section is to obtain Proposition 2.8 which is the key ingredient in the proof of Theorem 1.4.

Lemmas 2.1, 2.2, 2.3 are some well-known facts we need later.

Lemma 2.1. *Suppose that V is a nonzero D-stable subset of R.*

1. $\operatorname{Ann}(V) = \{ x \in R \mid xV = 0 \}$ *is a D-ideal.*
2. *If R is D-prime and $rV = 0$ for some $r \in R$, then $r = 0$.*

Proof. If $x \in \operatorname{Ann}(V), v \in V$ and $\delta \in D$, then $\delta(x)v = \delta(xv) - x\delta(v) = 0$ and hence $\delta(x) \in \operatorname{Ann}(V)$. It follows that $\operatorname{Ann}(V)$ is a D-ideal. Obviously, VR is a nonzero D-ideal and $\operatorname{Ann}(V)\cdot(VR) = 0$. Since R is D-prime, we get $\operatorname{Ann}(V) = 0$. So $r \in \operatorname{Ann}(V) = 0$. □

Lemma 2.2. *Let L be any nonzero Lie ideal of D. Then*

1. $g(L) = \sum_{\gamma \in L} \gamma(R)R$ *is a D-ideal of R.*
2. $\operatorname{Ann}_R(L) = \{ r \in R \mid rL = 0 \}$ *is a D-ideal of R.*
3. *If R is D-prime and $rL = 0$, then $r = 0$.*
4. *If R is D-prime, then D is not abelian.*

Proof. For (1), if $\delta \in D, \gamma \in L$ and $x \in R$, then

$$\delta\gamma(x) = [\delta, \gamma](x) + \gamma\delta(x) \in [\delta, \gamma](R) + \gamma(R)$$

and $\delta(\gamma(R)R) \subseteq \delta\gamma(R)R + \gamma(R)\delta(R) \subseteq g(L)$. It follows that $g(L)$ is a D-ideal of R.

For (2), if $x \in \operatorname{Ann}_R(L)$ and $\gamma \in L$, we get $x\gamma = 0$ and therefore we have $\delta(x)\gamma + x\delta\gamma = \delta(x\gamma) = 0$ for any $\delta \in D$. So

$$\delta(x)\gamma = -x\delta\gamma = -x[\delta, \gamma] - x\gamma\delta = 0$$

since L is a Lie ideal of D. Thus $\delta(x)L = 0$ and $\operatorname{Ann}_R(L)$ is a D-ideal of R.

For (3), simply note that $\operatorname{Ann}_R(L)\cdot g(L) = 0$. Since $L \neq 0$, we have $g(L) \neq 0$ and $r \in \operatorname{Ann}_R(L) = 0$ by D-primeness of R.

For (4), suppose that $[D, D] = 0$. For any $\delta, \gamma \in D$ and $x \in R$, we have $\delta(x)\gamma = [\delta, x\gamma] - x[\delta, \gamma] = 0$. It follows that $\delta(x)D = 0$ and by (3), we have $\delta(x) = 0$ for any $\delta \in D, x \in R$. We get $D = 0$ and this is a contradiction. So D cannot be abelian. □

Let $N(R) = \{\, r \in R \,|\, r^n = 0 \text{ for some } n \,\}$ be the nil radical of R. Furthermore, let $R^D = \{\, r \in R \,|\, \delta(r) = 0 \text{ for all } \delta \in D \,\}$ be the subring of D-constants of R. Note that any derivation of R in D is linear over R^D and cR is a D-ideal of R for any $c \in R^D$.

Lemma 2.3. *Suppose that R is D-prime. We have*

1. *If $c \in R^D$ and $rc = 0$ for some $r \in R$, then $r = 0$ or $c = 0$.*
2. *$R^D \cap N(R) = 0$.*
3. *If $a \in R, 0 \neq c \in R^D$ such that $ac \in R^D$, then $a \in R^D$.*
4. *If $\operatorname{char} R = 2$ and $a \in R$, then $a^2 \in R^D$.*

Proof. For (1), if $0 \neq c \in R^D$ such that $rc = 0$, then cR is a nonzero D-ideal and $r \in \operatorname{Ann}(cR) = 0$ by Lemma 2.1.

For (2), if $x \in R^D \cap N(R)$, then $xx \cdots x = 0$ and hence $x = 0$ by (1).

For (3), if δ is any element in D, $0 = \delta(ac) = \delta(a)c$ and hence $\delta(a) = 0$ by (1). Therefore $a \in R^D$.

For (4), if δ is any element in D, then $\delta(a^2) = 2a\delta(a) = 0$ since $\operatorname{char} R = 2$. It follows that $a^2 \in R^D$. □

We remark that by Lemma 2.3 (1), R^D is a domain, regular in R, and hence $\operatorname{char} R = 0$ or a prime $p > 0$.

Now we start to see what can happen when D is not prime.

Lemma 2.4. *Suppose that R is D-prime. If L, M are nonzero Lie ideals of D such that $[L, M] = 0$, then*

1. *$\alpha(x)\beta + \beta(x)\alpha = 0$ for any $x \in R, \alpha \in L, \beta \in M$.*
2. *$L \cap M \neq 0$.*

Proof. For (1), if $\alpha \in L, \beta \in M, \gamma \in D, x \in R$, then $[\alpha, x\gamma] \in L$ and $[\beta, [\alpha, x\gamma]] \in [M, L] = 0$. It follows that

$$\begin{aligned} 0 &= [\beta, [\alpha, x\gamma]] = [\beta, x[\alpha, \gamma] + \alpha(x)\gamma] \\ &= x[\beta, [\alpha, \gamma]] + \beta(x)[\alpha, \gamma] + \alpha(x)[\beta, \gamma] + \beta\alpha(x)\gamma \\ &= \beta(x)[\alpha, \gamma] + \alpha(x)[\beta, \gamma] + \beta\alpha(x)\gamma \end{aligned}$$

for any $x \in R$. Replacing x by xt, $t \in R$, we get

$$\begin{aligned} 0 &= \beta(x)t[\alpha, \gamma] + x\beta(t)[\alpha, \gamma] + \alpha(x)t[\beta, \gamma] + x\alpha(t)[\beta, \gamma] \\ &\quad + \beta\alpha(x)t\gamma + \alpha(x)\beta(t)\gamma + \beta(x)\alpha(t)\gamma + x\beta\alpha(t)\gamma \\ &= t(\beta(x)[\alpha, \gamma] + \alpha(x)[\beta, \gamma] + \beta\alpha(x)\gamma) \\ &\quad + x(\beta(t)[\alpha, \gamma] + \alpha(t)[\beta, \gamma] + \beta\alpha(t)\gamma) + (\alpha(x)\beta(t) + \beta(x)\alpha(t))\gamma \end{aligned}$$

and $(\alpha(x)\beta(t) + \beta(x)\alpha(t))D = 0$ for any $x, t \in R, \alpha \in L, \beta \in M$. Therefore, $\alpha(x)\beta + \beta(x)\alpha = 0$ for any $x \in R, \alpha \in L, \beta \in M$ by Lemma 2.2 (3).

For (2), suppose that $L \cap M = 0$. Note that for any $\alpha \in L, \beta \in M, x \in R$, $\alpha(x)\beta = [\alpha, x\beta] \in L$ and $\beta(x)\alpha = [\beta, x\alpha] \in M$. By (1), $\alpha(x)\beta = -\beta(x)\alpha \in L \cap M = 0$. This means $\alpha(x)M = 0$. Since $M \neq 0$, we have $\alpha(x) = 0$ for any $\alpha \in L, x \in R$ by Lemma 2.2 (3). This means that $L = 0$ and we have a contradiction. □

Next, we need a theorem of Jordan and we include its proof for the reader's convenience.

Theorem 2.5 (Theorem 1, [Jor78])**.** *Let $0 \neq J$ be a Lie ideal of D. If J is an R-submodule of D, then there exists a nonzero D-ideal I of R such that $ID \subseteq J$.*

Proof. Since $J \neq 0$, we know that $g(J) = \sum_{\gamma \in J} \gamma(R)R$ is a nonzero D-ideal of R and we want to show that $g(J)D \subseteq J$. For any $\gamma \in J$, since $\gamma(R)RD = \gamma(R)D$, we only need to show $\gamma(R)D \subseteq J$. For any $\delta \in D, \gamma \in J$ and $x \in R$,

$$\gamma(x)\delta = [\gamma, x\delta] - x[\gamma, \delta] \in J + RJ \subseteq J$$

and this yields the theorem. □

For a Lie ideal L of D, L may not be an R-submodule of D. Following [Jor00], we define $\tilde{L} = \{\, \delta \in D \mid R\delta \subseteq L \,\}$. Then $\tilde{L} \subseteq L$ is both an ideal and R-submodule of D.

Lemma 2.6. *Suppose that R is D-prime. If L, M are nonzero Lie ideals of D such that $[L, M] = 0$, then we have*

(1) *$\tilde{L} = 0$ or $\tilde{M} = 0$.*
(2) *$\alpha\beta = 0 = \beta\alpha$ for any $\alpha \in L, \beta \in M$.*

Proof. For (1), suppose by way of contradiction that $\tilde{L} \neq 0$ and $\tilde{M} \neq 0$. By Theorem 2.5, there exist nonzero D-ideals A, B of R such that $AD \subseteq \tilde{L}$ and $BD \subseteq \tilde{M}$. For any $\delta, \gamma \in D, a \in A, b \in B$, $[a\delta, b\gamma] \in [L, M] = 0$ and hence

$$ab[\delta, \gamma] + a\delta(b)\gamma - b\gamma(a)\delta = 0.$$

Since A is an ideal of R, replace a by ta and we get

$$\begin{aligned} 0 &= tab[\delta, \gamma] + ta\delta(b)\gamma - b\gamma(ta)\delta \\ &= t(ab[\delta, \gamma] + a\delta(b)\gamma - b\gamma(a)\delta) - b\gamma(t)a\delta \\ &= -b\gamma(t)a\delta \end{aligned}$$

for any $\delta, \gamma \in D$ and $a \in A, b \in B$. Thus $AB\gamma(t)D = 0$ for any $t \in R, \gamma \in D$ and it follows from Lemma 2.2 (3) that $AB\gamma(t) = 0$. Since A, B are nonzero D-ideals and R is D-prime, this yields $\gamma(t) = 0$ for any $\gamma \in D, t \in R$. Namely, $D = 0$ and this is a contradiction. Therefore, either $\tilde{L} = 0$ or $\tilde{M} = 0$.

For (2), let $J = L \cap M$. Then $[L, J] = [M, J] = 0$. For any $\gamma \in J, \alpha \in L, \beta \in M$, we have $[\alpha, \gamma] = [\beta, \gamma] = 0$ and hence

$$\beta\alpha(x)\gamma = [\beta, [\alpha, x\gamma]] \in [M, L] = 0$$

for any $x \in R$. It follows that $\beta\alpha(x)J = 0$. Since $J \neq 0$ by Lemma 2.4 (2), we get $\beta\alpha(x) = 0$ for any $x \in R$ by Lemma 2.2 (3). So $\beta\alpha = 0$ and similarly, we have $\alpha\beta = 0$. □

The following uses techniques from the proof of [Jor00, Lemma 2].

Lemma 2.7. *Suppose that R is D-prime and J is a Lie ideal of D. If $J \neq 0$, $[J,J]=0$ and $\tilde{J}=0$, then $\gamma(R) \subseteq R^D$ for any $\gamma \in J$.*

Proof. For any $\beta, \gamma \in J, \delta \in D$ and $a, b \in R$, we have $\gamma(a)\beta = [\gamma, a\beta] \in J$ since $[J,J]=0$. It follows that $[b\delta, \gamma(a)\beta] \in J$ and $[b\gamma(a)\delta, \beta] \in J$. So we get

$$[b\delta, \gamma(a)\beta] = b\gamma(a)[\delta,\beta] + b\delta\gamma(a)\beta - \gamma(a)\beta(b)\delta \in J$$

and

$$\begin{aligned}[b\gamma(a)\delta, \beta] &= b\gamma(a)[\delta,\beta] - \beta(b\gamma(a))\delta \\ &= b\gamma(a)[\delta,\beta] - \beta(b)\gamma(a)\delta - b\beta\gamma(a)\delta \in J.\end{aligned}$$

This shows $b\delta\gamma(a)\beta + b\beta\gamma(a)\delta = [b\delta, \gamma(a)\beta] - [b\gamma(a)\delta, \beta] \in J$ for any $b \in R$. Therefore,

$$\delta\gamma(a)\beta + \beta\gamma(a)\delta \in \tilde{J} = 0$$

and hence $\delta\gamma(a)\beta + \beta\gamma(a)\delta = 0$ for any $\beta, \gamma \in J, \delta \in D$ and $a \in R$. Now, $\beta, \gamma \in J$ implies that $\beta\gamma = 0$ by Lemma 2.6 (2). Therefore, $\delta\gamma(a)\beta = 0$ and $\delta\gamma(a)J = 0$ for any $\gamma \in J, \delta \in D, a \in R$. By Lemma 2.2 (3), we see that $\delta\gamma(a) = 0$ since $J \neq 0$ and it follows $\gamma(a) \in R^D$ for any $\gamma \in J, a \in R$. □

When D is not prime, we have the following

Proposition 2.8. *Suppose that R is D-prime and D is not prime. Then D contains a nonzero abelian ideal J such that $\gamma(R) \subseteq R^D$ for any $\gamma \in J$.*

Proof. Since D is not prime, there exist nonzero ideals L, M of D such that $[L,M]=0$. Let $J = L \cap M$. Then clearly $[J,J]=0$ and we have $J \neq 0$ by Lemma 2.4 (2). By Lemma 2.6 (1), $\tilde{J}=0$. Now we see that $\gamma(R) \subseteq R^D$ for any $\gamma \in J$ by Lemma 2.7. □

3. Localizations

We start this section by studying the special case when $D = R\delta \neq 0$ and $\operatorname{char} R = 2$. Note that in this case, we have $[x\delta, y\delta] = \delta(xy)\delta$ for any $x, y \in R$. In particular, $[D,D] = \delta(R)\delta$ is a Lie ideal of D.

Theorem 3.1. *Let R be D-prime, $D = R\delta \neq 0$ and $\operatorname{char} R = 2$. Then the Lie ring $R\delta$ is Lie prime if and only if $\delta(\delta(R)\delta(R)) \neq 0$.*

Proof. If $R\delta$ is prime, it is easy to prove that $\delta(\delta(R)\delta(R)) \neq 0$. Indeed, if $\delta(\delta(R)\delta(R)) = 0$, then for any $r, s \in R$,

$$[\delta(r)\delta, \delta(s)\delta] = \delta(\delta(r)\delta(s))\delta = 0.$$

By Lemma 2.2 (3), we get a nonzero Lie ideal $[D,D] = \delta(R)\delta$ of $D = R\delta$ such that $[\delta(R)\delta, \delta(R)\delta] = 0$. This contradicts the primeness of $R\delta$.

Conversely, assume that $D = R\delta$ is not prime. Then by Proposition 2.8, $R\delta$ has a nonzero abelian Lie ideal J such that $\gamma(R) \subseteq R^D$ for any $\gamma \in J$. Let $V = \{\, x \in R \mid x\delta \in J \,\}$. Note that if $x \in V$, then $\delta(x)\delta = [\delta, x\delta] \in J$ and hence $\delta(x) \in V$. So V is a δ-stable additive subgroup of R. Since

$0 \neq J \subseteq R\delta$, we see that $V \neq 0$. For any $v \in V$ we have $v\delta \in J$ and hence $v\delta(R) \subseteq R^D$. It follows that $\delta(V\delta(R)) = 0$.

Next, for any $v, w \in V$, $\delta(vw)\delta = [v\delta, w\delta] \in [J, J] = 0$. It follows that $\delta(vw)D = \delta(vw)R\delta = 0$ and hence $\delta(vw) = 0$ by Lemma 2.2 (3).

Now let $v, w \in V$ and $r, s \in R$. Then using $\delta(vw) = 0$, we have

$$\begin{aligned} vw\delta(\delta(r)\delta(s)) &= \delta(vw\delta(r)\delta(s)) = \delta(v\delta(r)w\delta(s)) \\ &= \delta(v\delta(r))w\delta(s) + v\delta(r)\delta(w\delta(s)) = 0, \end{aligned}$$

since $\delta(V\delta(R)) = 0$. Thus, $VV\delta(\delta(R)\delta(R)) = 0$ and hence

$$(RV)(RV)\delta(\delta(R)\delta(R)) = 0.$$

Since V is nonzero and δ-stable, RV is a nonzero D-stable ideal of R. By Lemma 2.1, we get $\delta(\delta(R)\delta(R)) = 0$. □

Now we obtain our characterization of the primeness of the Lie ring D by using localizations.

First we consider localizations of the rings. Let $C = R^D$ be the ring of D-constants of R. Since R is D-prime, it follows that elements of C^* are regular in R by Lemma 2.3 (1). Let $K = C^{-1}C$ be the field of fractions of C and let $S = C^{-1}R$ be the localization of R at C^*. Then S is an associative algebra with 1 over K. Note that $\dim_K S \geq 2$. Indeed, if $\dim_K S = 1$, then $S = K$ and Lemma 2.3 (3) implies that $R = C$ and $\delta(R) = 0$ for any $\delta \in D$. This means $D = 0$ and we get a contradiction.

Next we consider derivations. Each derivation δ of R in D extends uniquely to a derivation $\hat{\delta}$ of S given by $\hat{\delta}(x/c) = \delta(x)/c$ where $x \in R$ and $c \in C^*$. This is well-defined since δ is C-linear. Then it is obvious that $\hat{D} = \{\, \hat{\delta} \mid \delta \in D \,\}$ is a Lie subring of $\mathrm{Der}(S)$ and that $\hat{D} \cong D$ as Lie rings and as R-modules. Furthermore, since $K = C^{-1}C$ and every derivation of $\hat{D}$ is K-linear, we have $\hat{D} \subseteq \mathrm{Der}_K(S)$. Let $\bar{D} = K\hat{D} = C^{-1}\hat{D}$ and note that a typical element in $\bar{D}$ has the form $c^{-1}\hat{\delta}$ where $\delta \in D$ and $c \in C^*$. Then $\bar{D}$ is a Lie subalgebra of $\mathrm{Der}_K(S)$ and also an S-submodule of $\mathrm{Der}_K(S)$. Now it is straight forward to check that $\bar{D}$ is a prime Lie ring if and only if $\bar{D}$ is a prime Lie algebra over K. Furthermore, this occurs if and only if $\hat{D}$ is a prime Lie ring and hence if and only if D is a prime Lie ring. Finally, since R is D-prime, it follows that S is $\bar{D}$-prime.

With these notations, we are able to prove our main theorems.

Theorem 3.2. *Suppose that R is D-prime. Then*

1. *D is Lie prime except possibly when $\operatorname{char} R = 2$ and $\bar{D}$ is cyclic as an S-module.*
2. *If $\operatorname{char} R = 2$ and $\bar{D} = S\theta$ is cyclic as an S-module, then D is Lie prime if and only if $\theta(\theta(S)\theta(S)) \neq 0$.*

Proof. For (1), suppose that D is not prime. By Proposition 2.8, there exists a nonzero ideal J of D such that $\gamma(R) \subseteq R^D = C$ for any $\gamma \in J$. Since $J \neq 0$, there exists some nonzero $\gamma \in J$. Then $\gamma(a) \neq 0$ for some $a \in R$

and we see that $a \notin C$. Furthermore, $2\gamma(a)a = \gamma(a^2) \in C$. If $\operatorname{char} R \neq 2$, then $2\gamma(a) \in C$ is nonzero and hence $a \in C$ by Lemma 2.3 (3). This is a contradiction and we conclude that $\operatorname{char} R = 2$.

Since $\gamma(R) \subseteq C$, we have $\delta\gamma(x) = 0$ for any $x \in R, \delta \in D$. Expanding $\delta\gamma(ax) = 0$, we get

$$\gamma(a)\delta(x) + \delta(a)\gamma(x) = \delta\gamma(a)x + \gamma(a)\delta(x) + \delta(a)\gamma(x) + a\delta\gamma(x) = 0$$

for any $x \in R, \delta \in D$. Therefore $\gamma(a)\delta = \delta(a)\gamma$ and hence $\gamma(a)\hat{\delta} = \delta(a)\hat{\gamma}$. Since $\gamma(a) \in C^*$, it follows that $\hat{\delta} \in S\hat{\gamma}$ for any $\delta \in D$. We get that $\hat{D} \subseteq S\hat{\gamma}$ and hence $\bar{D} \subseteq S\hat{\gamma}$. Since $\bar{D}$ is an S-module and $\hat{\gamma} \in \bar{D}$, we see that $S\hat{\gamma} \subseteq \bar{D}$ and $\bar{D} = S\hat{\gamma}$. Therefore, if $\operatorname{char} R \neq 2$ or if $\bar{D}$ is not cyclic as an S-module, then D is Lie prime.

For (2), suppose that $\operatorname{char} R = 2$ and $\bar{D} = S\theta$ is cyclic as an S-module. As we observed, D is prime if and only if $\bar{D}$ is prime. Since S is $\bar{D}$-prime, Theorem 3.1 clearly yields the result. □

Theorem 3.3. *Let R be D-prime. Then D is Lie prime if and only if* $\operatorname{char} R \neq 2$ *or* $\dim_K S \neq 2$. *Moreover, when D is not Lie prime, $\bar{D}$ is the unique nonabelian Lie algebra of K-dimension* 2.

Proof. Suppose that D is not prime. Following the proof of Theorem 3.2, we see that $\operatorname{char} R = 2$ and there exist $\gamma \in D, a \in R$ such that $\gamma(a) \neq 0$ and $\gamma(R) \subseteq C$.

Now for any $x \in R$, $x\gamma(a) = \gamma(xa) - \gamma(x)a \in Ca + C$ since $\gamma(R) \subseteq C$. It follows that $x \in Ka + K$ for any $x \in R$ and hence $S \subseteq Ka + K$. Since $\dim_K S \geq 2$, we see that $\dim_K S = 2$ and $S = Ka + K$. Therefore, if $\operatorname{char} R \neq 2$ or if $\dim_K S \neq 2$, then D is Lie prime.

Conversely, suppose $\operatorname{char} R = 2$ and $\dim_K S = 2$. We want to show that D is not prime. Since $\dim_K S = 2$, choose $a \in S, a \notin K$ such that $S = Ka + K$. Define the map $\phi\colon \bar{D} \to S$ by $\phi(c^{-1}\hat{\delta}) = \hat{\delta}(a)/c$. It is straight forward to check that ϕ is well-defined and K-linear. Furthermore, if $\phi(c^{-1}\hat{\delta}) = 0$, then $\hat{\delta}(a) = 0$ and $\hat{\delta}(S) = 0$ since $\hat{\delta}$ is K-linear and $S = Ka + K$. It follows that $c^{-1}\hat{\delta} = 0$ and thus ϕ is one-to-one. This implies that $\dim_K \bar{D} \leq \dim_K S = 2$. Since R is D-prime, Lemma 2.2 (4) implies that D is not abelian and hence neither is $\bar{D}$. So $\dim_K \bar{D} \neq 1$ and $\bar{D}$ is the unique nonabelian Lie algebra of K-dimension 2. This Lie algebra has a 1-dimensional ideal $[\bar{D}, \bar{D}]$ and hence $\bar{D}$ is not prime. Therefore, D is not prime. □

This, of course, proves Theorem 1.4.

4. Without Localizations

In this section, we avoid localizations and give the characterizations of the primeness of D entirely within R. This affords us the opportunity to offer slightly different versions of some of the arguments used in Section 3. We first consider conditions on D that give this characterization. Again, let $C = R^D$ be the subring of D-constants of R. For convenience, we say that

D is almost cyclic if there exist some nonzero $c \in C$ and $\gamma \in D$ such that $cD \subseteq R\gamma$. Then we have

Theorem 4.1. *Suppose that R is D-prime.*

(1) *D is Lie prime except possibly when* $\operatorname{char} R = 2$ *and D is almost cyclic.*
(2) *If* $\operatorname{char} R = 2$ *and D is almost cyclic with $cD \subseteq R\gamma$, then D is Lie prime if and only if $\gamma(\gamma(R)\gamma(R)) \neq 0$.*

Proof. For (1), suppose that D is not Lie prime. Following the proof of Theorem 3.2, we get $\operatorname{char} R = 2$ and there exist $\gamma \in D, a \in R$ such that $0 \neq c = \gamma(a) \in C$, $\gamma(R) \subseteq C$ and $\gamma(a)\delta = \delta(a)\gamma$ for any $\delta \in D$. Then we have $cD \subseteq R\gamma$, so D is almost cyclic.

For (2), suppose that $\operatorname{char} R = 2$ and D is almost cyclic with $cD \subseteq R\gamma$ for some nonzero $c \in C$ and $\gamma \in D$. Recall that when $\operatorname{char} R = 2$, we have $[x\gamma, y\gamma] = \gamma(xy)\gamma$ for any $x, y \in R$.

If $\gamma(\gamma(R)\gamma(R)) = 0$, then

$$c^2 D^{(1)} = c^2[D, D] = [cD, cD] \subseteq [R\gamma, R\gamma] = \gamma(R)\gamma$$

and

$$c^4 D^{(2)} = [c^2 D^{(1)}, c^2 D^{(1)}] \subseteq [\gamma(R)\gamma, \gamma(R)\gamma] \subseteq \gamma(\gamma(R)\gamma(R))\gamma = 0.$$

So $c^4 D^{(2)}(R) = 0$ and $D^{(2)} = 0$ by Lemma 2.3 (1). But Lemma 2.2 (4) tells us that $D^{(1)} \neq 0$. It follows that D is not prime.

Conversely, suppose that D is not prime. Following the proof of part (1), there exist nonzero $b \in C$ and $\beta \in D$ such that $\beta(R) \subseteq C$ and $bD \subseteq R\beta$. So we have

$$[bD, bD] \subseteq [R\beta, R\beta] = \beta(R)\beta$$

and $b^2[D, D](R) \subseteq \beta(R)\beta(R) \subseteq C$. So $[D, D](R) \subseteq C$ by Lemma 2.3 (3). Now, since $\gamma \in D$ and $R\gamma \subseteq D$, we see that $\gamma(R)\gamma = [R\gamma, R\gamma] \subseteq [D, D]$ and hence $\gamma(R)\gamma(R) \subseteq [D, D](R) \subseteq C$. It follows that $\gamma(\gamma(R)\gamma(R)) = 0$ and the proof is complete. □

Next we consider conditions on R that give us a characterization for the primeness of D. For convenience, we say that R is 2-dimensional if there exist $a \in R \setminus C$ and a nonzero element $c \in C$ such that $T = Ca + C$ is a subring of R and $cR \subseteq T \subseteq R$. Then we have

Theorem 4.2. *Suppose that R is D-prime. Then D is Lie prime if and only if* $\operatorname{char} R \neq 2$ *or R is not 2-dimensional. Moreover, when D is not Lie prime, then D is solvable of derived length 2.*

Proof. Suppose that D is not Lie prime. Following the proof of Theorem 3.3, we get $\operatorname{char} R = 2$ and there exist $\gamma \in D, c, a \in R$ such that $0 \neq c = \gamma(a) \in C$ and $x\gamma(a) \in Ca + C$ for any $x \in R$. It follows that $cR \subseteq Ca + C \subseteq R$. Since $\operatorname{char} R = 2$, Lemma 2.3 (4) implies that $a^2 \in C$ and hence $T = Ca + C$ is a subring of R. This says that R is 2-dimensional. Therefore, if $\operatorname{char} R \neq 2$ or if R is not 2-dimensional, then D is Lie prime.

Conversely, suppose that $\operatorname{char} R = 2$ and that R is 2-dimensional. Then there exist $a \in R \setminus C$ and $0 \neq c \in C$ such that $T = Ca + C$ is a subring of R and $cR \subseteq T \subseteq R$. Since $c \in C$ and since any derivation in D is C-linear, we have $\delta(cR) = c\delta(R) \subseteq T$ for any $\delta \in D$. So for any $\delta, \gamma \in D$, we can write $\delta(ca) = ua + v$ and $\gamma(ca) = xa + y$ for some $u, v, x, y \in C$. Then

$$\delta\gamma(c^2a) = \delta(c\gamma(ca)) = \delta(c(xa + y)) = x\delta(ca) + \delta(cy) = x(ua + v)$$

and similarly, $\gamma\delta(c^2a) = u(xa+y)$. It follows that $[\delta, \gamma](c^2a) = xv - uy \in C$. So $c^2[\delta, \gamma](a) \in C$ and $[\delta, \gamma](a) \in C$ by Lemma 2.3 (3). Since $cR \subseteq Ca + C$, we see that $c[\delta, \gamma](R) = [\delta, \gamma](cR) \subseteq C$ and $[\delta, \gamma](R) \subseteq C$ by Lemma 2.3 (3) again. We have shown that $[\delta, \gamma](R) \subseteq C$ for any $\delta, \gamma \in D$. This implies that for any $\alpha, \beta \in [D, D] = D^{(1)}$, we have $\alpha(R) \subseteq C$ and $\beta(R) \subseteq C$. Thus $\alpha\beta = 0 = \beta\alpha$ and $[\alpha, \beta] = 0$. It follows that $D^{(2)} = [D^{(1)}, D^{(1)}] = 0$. However, $D^{(1)} \neq 0$ by Lemma 2.2 (4). We conclude that D is solvable of derived length 2 and hence D is not prime. □

Suppose R is 2-dimensional, so that $cR \subseteq Ca + C \subseteq R$ for some nonzero element $c \in C$ and element $a \in R \setminus C$. We note that if c is a unit in R, then $cR = R$ and hence $R = Ca + C$. In particular, this occurs when C is a field and hence when R is D-simple.

5. Examples

Finally, we offer several examples, mostly related to the characteristic 2 exceptions. We start by considering the number of generators of the R-module D. For convenience, we say that D is precisely n-generated if D can be generated as an R-module by n elements, but by no fewer number.

Example 5.1. We first find R and D such that R is D-prime, D is prime and yet D is not cyclic as an R-module. To this end, let F be a field of characteristic 2, take $R = F(x_1, x_2, \dots, x_n)$ to be the rational function field over F in n variables and let D be the left R-module generated by $\partial/\partial x_i$ for $i = 1, 2, \dots, n$. Since R is a field, it is obviously D-simple and D-prime. Moreover, D is a vector space over R of dimension n. In particular, when $n \geq 2$, D is not cyclic as an R-module and Theorem 1.1 implies that D is Lie simple and hence Lie prime. In this way, we get a precisely n-generated D which is prime.

Next, we show that an almost cyclic D need not be cyclic.

Example 5.2. Let F be a field of characteristic 2 and let $C = F[t_1, \dots, t_n]$ be the polynomial ring in $n \geq 1$ variables over F. Define $R = C[x|x^2 = 0]$. Thus any element u in R has the unique form $u = u_0 + u_1x$ with $u_0, u_1 \in C$. Let $\gamma_j = t_j{\cdot}\partial/\partial x, \alpha = x{\cdot}\partial/\partial x$ and set $D = \sum_{j=1}^{n} R\gamma_j + R\alpha$. It is obvious that $R^D = C$ and that the elements of C^* are regular in R.

If A is a nonzero D-stable ideal of R, then $A \cap C \neq 0$. Indeed, if $0 \neq u = u_0 + u_1 x \in A$, then either $u \in C^*$ or $\gamma_1(u) \in C^*$. Thus every nonzero D-stable ideal of R contains a regular element and R is D-prime. In particular, since R is clearly 2-dimensional, Theorem 4.2 implies that D is not prime.

Finally, note that $D = I{\cdot}(\partial/\partial x)$, where I is the ideal of R generated by $T = \{ t_1, \ldots, t_n, x \}$. Furthermore, evaluation at x yields an R-module isomorphism from D to I. Since the elements of T are $F = R/I$ linearly independent modulo I^2, it follows that I, and hence D, is precisely $(n+1)$-generated as an R-module.

The preceding two examples show, when $\operatorname{char} R = 2$ and R is D-prime, that there exist n-generator D that are prime and also n-generator D that are not prime. It follows that the number of R-generators of D cannot by itself determine whether D is prime or not.

At this point, we consider the structure of a 2-dimensional ring in characteristic 2 when its nil radical is not zero. For convenience, if C is any integral domain with field of fractions K, we say that C is 2-integrally closed if $k \in K$ with $k^2 \in C$ implies that $k \in C$. Obviously, if C is a unique factorization domain, then C is integrally closed and hence 2-integrally closed, so this property occurs reasonably often.

Lemma 5.3. *Let R be D-prime, set $C = R^D$ and assume that R is a 2-dimensional ring in characteristic 2.*

(1) *If R is the direct sum $R = C + I$ with $I \lhd R$, then $I = N(R)$ is the nil radical of R. Furthermore, as C-modules, I is isomorphic to an ideal of C.*
(2) *If $N(R) \neq 0$ and C is 2-integrally closed, then R is the direct sum $R = C + N(R)$.*

Proof. For (1), let $x \in I$. Then the direct sum decomposition implies that $x^2 \in I \cap C = 0$, so I must consist of elements of R of square 0 and thus $I \subseteq N(R)$, the nil radical of R. But $R/I \cong C$ is a domain, so we see that $I = N(R)$. Now let $a \in R \setminus C$ and let $0 \neq c \in C$ be given by the fact that R is 2-dimensional. Then $cR \subseteq C + Ca \subseteq R$, where $C + Ca$ is a direct sum by Lemma 2.3 (3), and we let $\pi \colon C + Ca \to Ca$ be the natural C-module projection. Since cI is a nil ideal of $C + Ca$, it is disjoint from C, and therefore π is one-to-one on cI. As C-modules we clearly have $I \cong cI$, $cI \cong \pi(cI)$, $\pi(cI) \subseteq Ca$ and $Ca \cong C$. Thus I is C-isomorphic to a C-submodule of C, namely an ideal of C.

For (2), fix any $0 \neq a \in N(R)$ and set $K = C^{-1}C$. If $S = C^{-1}R$ is the localization of R at the nonzero elements of C, then S is a 2-dimensional K-algebra and hence $R \subseteq S = K + Ka$. Let $r = k_1 + k_2 a \in R$ with $k_1, k_2 \in K$. Then, by Lemma 2.3 (2)(4), $r^2 \in C$ and $r^2 = k_1^2$ since $a^2 = 0$. Thus $k_1^2 \in C$ and, since C is 2-integrally closed, it follows that $k_1 \in C$. We have therefore shown that $C \subseteq R \subseteq C + Ka$ and hence $R = C + (R \cap Ka) = C + N(R)$, where the latter is a direct sum since $K + Ka$ is direct. $\square$

We now construct an example to show that the splitting in the preceding lemma does not always occur.

Example 5.4. Let F be a field of characteristic 2 and consider the polynomial ring $F[x]$. If $f(x) \in F[x]$, we write $f(x) = \sum_i f_i x^i$ so that $f_i \in F$ is the coefficient of x^i. We take $C = \{\, f(x) \in F[x] \mid f_1 = 0 \,\}$. It is easy to see that C is the F-subalgebra of $F[x]$ generated by 1, x^2 and x^3. In particular, $x^3/x^2 = x$ belongs to the field of fractions K of C. Furthermore, $x \in K \setminus C$ and $x^2 \in C$, so C is not 2-integrally closed.

Let T be the ring $F[x,y]/(y^2)$ so that $T = F[x] + F[x]a$ with $a^2 = 0$, and let $R \subseteq T$ be the set of all $f(x) + g(x)a$ with $g_1 = 0$ and $f_1 = g_0$. Note that $\delta = \partial/\partial a$ is a derivation of T and it is easy to see that R is δ-stable with constants $R^\delta = F[x] \cap R = C$. We now check that R is a subalgebra of T. First, it is clearly closed under addition and scalar multiplication. Next, let $f(x)+g(x)a$ and $u(x)+v(x)a$ both belong to R. Then $g_1 = v_1 = 0$, $f_1 = g_0$ and $u_1 = v_0$. Also, using $a^2 = 0$, we have

$$[f(x) + g(x)a][u(x) + v(x)a] = fu(x) + (fv + ug)(x)a,$$

where the above are polynomial products, not composition. Since $g_1 = v_1 = 0$ and F has characteristic 2, we obtain

$$(fv + ug)_1 = f_1 v_0 + u_1 g_0 = g_0 v_0 + v_0 g_0 = 0.$$

Furthermore,

$$(fu)_1 = f_0 u_1 + f_1 u_0 = f_0 v_0 + g_0 u_0 = (fv + ug)_0,$$

and R is indeed closed under multiplication.

As in Example 5.2, it is easy to check that R is D-prime, where we set $D = R\delta$. First note that every element of C^* is regular in T and hence in R. Next, if B is a nonzero D-stable ideal of R, then $B \cap C \neq 0$. Indeed, if $0 \neq r = f(x) + g(x)a \in B$, then either $r \in C^*$ or $\delta(r) \in C^*$. Thus every nonzero D-stable ideal of R contains a regular element and R is D-prime. It is now clear that R is a 2-dimensional ring in characteristic 2.

In conclusion, if $R = C + I$ with $I \triangleleft R$, then by Lemma 5.3 (1),

$$I = N(R) = R \cap N(T) = R \cap F[x]a = \{\, g(x)a \mid g_0 = g_1 = 0 \,\}.$$

Hence

$$R = C + I = \{\, f(x) + g(x)a \mid f_1 = g_0 = g_1 = 0 \,\}.$$

But this is not the case, since $x + a \in R$, while $x + a$ does not belong to the above right hand side.

Finally, given the obvious analogy between Theorem 3.2 for the primeness of D and Theorem 1.1 for the simplicity of D, it is reasonable to ask whether there is a simplicity analog of Theorem 3.3. Surprisingly, this is not the case.

Example 5.5. Let F be a field of characteristic $p > 0$ and let R be the rational function field $R = F(x_1, x_2, \ldots, x_n)$ over F in n variables. Let δ be the F-derivation of R given by $\delta(x_i) = x_i^{p+1}$. If $K = F(x_1^p, x_2^p, \ldots, x_n^p)$, then K is certainly in the field of δ-constants R^δ. Furthermore, $\dim_K R = p^n$ with a basis given by all monomials $\mu = \prod_i x_i^{a_i}$ with $0 \le a_i \le p-1$. Since $x_i^p \in K$, it follows that each such monomial μ is a K-eigenvector for the operator δ with eigenvalue given by $\sum_i a_i x_i^p \in K$. Thus R has a basis of eigenvectors for δ and the only zero eigenvalue occurs when $a_1 = a_2 = \cdots = a_n = 0$. Thus we see that $R^\delta = K$ and hence that $\dim_{R^\delta} R = p^n$.

Now let $p = 2$ and take $D = R\delta$. Then the field R is D-simple, but δ is not onto, and hence Theorem 1.1 implies that D is not Lie simple. In other words, this is one of the exceptional characteristic 2 situations in the simplicity problem. But there is no bound for $\dim_{R^D} R = 2^n$, so there is no obvious simplicity analog for Theorem 3.3.

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