SEMIPRIMITIVITY OF GROUP ALGEBRAS: A SURVEY

D. S. Passman

University of Wisconsin-Madison

Abstract. Let K be a field and let G be a multiplicative group. The group ring $K[G]$ is an easily defined, rather attractive algebraic object. As the name implies, its study is a meeting place for two essentially different algebraic disciplines. Indeed, group ring results frequently require a blend of group theoretic and ring theoretic techniques. A natural, but surprisingly elusive, group ring problem concerns the semiprimitivity of $K[G]$. Specificially, we wish to find necessary and sufficient conditions on the group G for its group algebra to have Jacobson radical equal to zero. More generally, we wish to determine the structure of the ideal $JK[G]$. In the case of infinite groups, this problem has been studied with reasonable success during the past 40 years, and our goal here is to survey what is known. In particular, we describe some of the techniques used, discuss a number of the results which have been obtained, and mention several tantalizing conjectures.

§1. Introduction

Let us first consider a possible way of defining the polynomial ring in two variables, say x and y, over a field K. To start with, form the set $S = \{\, x^a y^b \mid a, b = 0, 1, 2, \dots \}$ of monomials in x and y, and define multiplication in S by $x^a y^b \cdot x^c y^d = x^{a+c} y^{b+d}$. In this way, S becomes an associative semigroup with identity element $1 = x^0 y^0$. Next, let $K[x, y] = K[S]$ be the K-vector space with basis consisting of the elements of S. In other words, every element of $K[x, y]$ is a formal finite sum $\sum k_{a,b} x^a y^b$ with coefficients $k_{a,b} \in K$. Of course, the addition in $K[x, y]$ is the usual vector space addition, and multiplication in $K[x, y]$ is defined distributively using the multiplication in S. Since the associative law for multiplication in S clearly carries over to $K[S]$, it follows that $K[x, y]$ is an associative K-algebra. Similarly, we could define the Laurent polynomial ring $K[x, y, x^{-1}, y^{-1}]$

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX by taking S to be the multiplicative group $S = \{\, x^a y^b \mid a, b = 0, \pm 1, \pm 2, \dots \,\}$ and again forming $K[S]$. Indeed, this is our first example of a group ring.

More generally, let K be a field and let G be any multiplicative group. Then the *group algebra* or *group ring* $K[G]$ is a K-vector space with basis consisting of the elements of G. Thus every element of $K[G]$ is a formal finite sum

$$\alpha = \sum_{g \in G} k_g g$$

with coefficients $k_g \in K$. Again, addition in $K[G]$ is the obvious vector space operation, and we define multiplication distributively using the given multiplication of G. In this way, $K[G]$ becomes an associative K-algebra, with structure highly dependent on the nature of G. Basic references for group algebras include the books [MZ73], [iP79a], [dP72a], [dP77a], [sS78a] and [sS93a].

As is well known, group rings are important tools in both group theory and ring theory. For example, they provide the correct framework to study and understand the ordinary and modular character theory of finite groups. Furthermore, when G is a polycyclic-by-finite group, then $K[G]$ is a right and left Noetherian K-algebra and hence it is a useful testing ground for the rich theory of noncommutative Noetherian rings. In turn, the module theory of the latter group algebra can feed back into group theory to yield information on the structure of abelian-by-polycyclic groups. But, group rings are more than just useful tools. They are easily defined, rather attractive algebraic objects which are worthy of being considered in their own right. Their study is necessarily ring theoretic in nature, but the techniques and proofs exhibit a strong group theoretic flavor. The goal of this paper is to survey the progress made on a rather ellusive group ring problem.

If R is an associative ring with 1, then a (right) *R-module* V is just a right R-vector space. Thus V is an additive abelian group which admits right multiplication by R, and such that this scalar multiplication satisfies the usual axioms. Of course, these rules are precisely equivalent to the existence of a natural ring homomorphism $\theta_V \colon R \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is the ring of endomorphisms of the additive abelian group V. We say that $V \neq 0$ is *irreducible* if V has no proper R-submodule. In other words, the irreducible R-modules are the natural analogs of the 1-dimensional vector spaces over fields. For convenience, we let $\mathrm{Irr}(R)$ denote the set of all such irreducible R-modules.

A ring R is said to be *primitive* if it has a faithful irreducible module. In other words, R is primitive if there exists $V \in \mathrm{Irr}(R)$ with θ_V a one-to-one map. Such rings have a nice, rather natural structure; they are dense sets of linear transformations over division rings. Unfortunately, primitive rings are fairly scarce, so the next best situation is to study the ring R by looking at all its irreducible modules. But there is still a fundamental obstruction here, namely

$$JR = \bigcap_{V \in \mathrm{Irr}(R)} \ker \theta_V = \{\, r \in R \mid Vr = 0 \text{ for all } V \in \mathrm{Irr}(R) \,\}.$$

This characteristic ideal is called the *Jacobson radical* of R, and we say that R is *semiprimitive* precisely when $JR = 0$. Thus R is semiprimitive if and only if it is a subdirect product of primitive rings. In particular, such rings are reasonably well understood.

It is therefore of some interest and importance to determine those groups G with semiprimitive group algebras $K[G]$. More generally, we should describe the structure of the Jacobson radical $JK[G]$ for any group G. In the case of finite groups, the semiprimitivity problem has a classical solution, dating from 1898, and depends only on the characteristic of the field. Specifically, we have

Theorem 1.1 (Maschke). [hM98] *Let G be a finite group and let K be a field.*

i. *If $\operatorname{char} K = 0$, then $K[G]$ is semiprimitive.*
ii. *If $\operatorname{char} K = p > 0$, then $K[G]$ is semiprimitive if and only if G has no elements of order p.*

Of course, the goal now is to extend this result, or some variant of it, to the case of infinite groups.

§2. Fields of Characteristic 0

It is not surprising that progress on the semiprimitivity problem for infinite groups began with fields of characteristic 0, and indeed with the field C of complex numbers. The first significant result appeared in 1950, with a proof using analytic methods, including the spectral norm and the auxiliary norm of $C[G]$.

Theorem 2.1. [cR50] *If C is the field of complex numbers, then every group algebra $C[G]$ is semiprimitive.*

This result intrigued a number of ring theorists who rightly felt that it should have an algebraic proof. Thus, for example, the semiprimitivity problem for fields of characteristic 0 appeared in the Ram's Head Inn problem list [iK57] (see also [iK70]), and an algebraic argument for Theorem 2.1 was quickly discovered. It is instructive to consider some of the ingredients of this new proof. Recall that an ideal I of any ring R is said to be *nil* if all elements of I are nilpotent. Since every nil ideal of R is contained in JR, a first step in proving that $K[G]$ is semiprimitive might be to show that it has no nonzero nil ideal. In this direction we have

Lemma 2.2. *Let K be a subfield of the complex numbers which is closed under complex conjugation. If G is any group, then $K[G]$ has no nonzero nil ideal.*

Proof. Define a map ${}^*\colon K[G] \to K[G]$ by

$$\Big(\sum_g k_g g\Big)^* = \sum_g \bar{k}_g g^{-1}$$

where $^-$ indicates complex conjugation. It is easy to see that $(\alpha\beta)^* = \beta^*\alpha^*$, $\alpha^{**} = \alpha$, and $(\alpha+\beta)^* = \alpha^* + \beta^*$. Furthermore, if $\alpha = \sum_g k_g g$, then the identity coefficient of $\alpha\alpha^*$ is equal to $\sum_g k_g \bar{k}_g = \sum_g |k_g|^2$. Hence $\alpha\alpha^* = 0$ if and only if $\alpha = 0$.

Let I be a nonzero ideal of $K[G]$ and choose $0 \neq \alpha \in I$. Then, by the above, $\beta = \alpha\alpha^*$ is a nonzero element of I, and β is easily seen to be $*$-symmetric. In other words, any nonzero ideal of $K[G]$ contains a nonzero $*$-symmetric element. Next, we claim that 0 is the unique $*$-symmetric nilpotent element. Indeed, if γ is $*$-symmetric and nilpotent, then so is any power of γ. Thus it suffices to assume that $\gamma^2 = 0$. But then $0 = \gamma^2 = \gamma\gamma^*$, so $\gamma = 0$ as required, and the result follows immediately from the latter two observations. $\square$

The second ingredient holds over any field. Note that if H is a subgroup of G, then $K[H]$ is naturally embedded in $K[G]$. Indeed, this is just the group ring analog of the obvious polynomial ring inclusion $K[x] \subseteq K[x,y]$. Furthermore, since $K[G]$ is a free right and left $K[H]$-module, using coset representatives for H in G as a free basis, we have

Lemma 2.3. *Let K be any field and let H be a subgroup of G.*

i. *If W is an irreducible $K[H]$-module, then there exists an irreducible $K[G]$-module V with W a submodule of V_H, the restriction of V to $K[H]$.*
ii. *$JK[G] \cap K[H] \subseteq JK[H]$.*

The remainder of the argument is of less interest. To start with, the Hilbert Nullstellensatz asserts that if A is a finitely generated commutative algebra over a field K, then JA is a nil ideal. Furthermore, recall that there is a trivial proof of this result in case K is nondenumerable. Indeed, the same proof shows, without the commutativity assumption, that if A is a countable dimensional algebra over a nondenumerable field, then JA is nil. In particular, it follows from this and Lemma 2.2 that if H is a countable group, then the complex group algebra $C[H]$ is semiprimitive. Finally, if G is any group and if $\alpha \in JC[G]$, then there exists a finitely generated and hence countable subgroup H of G with $\alpha \in C[H]$. But then Lemma 2.3(ii) yields

$$\alpha \in C[H] \cap JC[G] \subseteq JC[H] = 0,$$

and Theorem 2.1 is proved.

Much more important is the later work of Amitsur on the behavior of the radical under field extensions. If A is a K-algebra and if F is a field containing K, then we denote the F-algebra $F \otimes_K A$ by A^F. Thus A^F is the largest ring generated by its commuting subrings F and A, with the two copies of K identified.

Theorem 2.4. [sA57] *Let $F \supseteq K$ be fields and let A be a K-algebra.*

i. *$J(A^F) \cap A \subseteq JA$ with equality when F/K is algebraic.*

ii. *If F/K is a finite separable extension, then $J(A^F) = F \otimes_K JA$.*
iii. *If F is a nontrivial purely transcendental extension of K, then $J(A^F) = F \otimes_K I$ for some nil ideal I of A.*

Since $K[G]^F = F \otimes_K K[G] = F[G]$, the preceding result and Lemma 2.2 applied to the field Q of rational numbers yield

Theorem 2.5 (Amitsur). [sA59] *Let K be a field of characteristic 0 so that K contains the rational numbers Q, and let G be an arbitrary group.*
i. *If K/Q is not algebraic, then $K[G]$ is semiprimitive.*
ii. *If K/Q is algebraic, then $JK[G] = K \otimes_Q Q[G]$ and $K[G]$ has no nonzero nil ideal.*

In particular, the semiprimitivity problem for algebraic extensions of Q reduces to Q itself. Presumably $Q[G]$ is always semiprimitive, but unfortunately the above result marks the extent of our knowledge. There has been no significant progress on the characteristic 0 problem since Theorem 2.5 appeared in 1959.

§3. Fields of Characteristic $p > 0$

Now let us turn to modular fields and assume for the remainder of this paper that $\operatorname{char} K = p > 0$. In view of Theorem 1.1, it is reasonable to suppose $K[G]$ is semiprimitive if and only if G is a p'-*group*, that is a group with no elements of order p. One direction of this is most likely true, but as we will see, the other direction is decidedly false. We begin with an interesting trace argument.

For any group G, let $\operatorname{tr}\colon K[G] \to K$ be the map which reads off the identity coefficient, so that $\operatorname{tr}(\sum k_g g) = k_1$. Then tr is obviously a K-linear functional, and it is easy to see that $\operatorname{tr}\alpha\beta = \operatorname{tr}\beta\alpha$ for all $\alpha, \beta \in K[G]$. Next, we note that if A is any K-algebra and if $\alpha_1, \alpha_2, \dots, \alpha_s \in A$, then

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_s)^{p^n} = \alpha_1^{p^n} + \alpha_2^{p^n} + \cdots + \alpha_s^{p^n} + \beta$$

for some $\beta \in [A, A]$, where the latter subspace is the span of all Lie products $[\gamma, \delta] = \gamma\delta - \delta\gamma$ with $\gamma, \delta \in A$.

Lemma 3.1. *If G is a p'-group, then $K[G]$ has no nonzero nil ideal.*

Proof. Suppose $\alpha = \sum k_g g \in K[G]$ is nilpotent, and choose n sufficiently large so that $\alpha^{p^n} = 0$. Then by the preceding formula,

$$0 = \alpha^{p^n} = \sum_{g \in G} (k_g)^{p^n} g^{p^n} + \beta$$

for some $\beta \in \big[K[G], K[G]\big]$. In particular, since tr annihilates all Lie products, we have $\operatorname{tr}\beta = 0$ and hence

$$0 = \sum_{g \in G} (k_g)^{p^n} \operatorname{tr} g^{p^n}.$$

But G is a p'-group, so $g^{p^n} = 1$ if and only if $g = 1$, and therefore $\operatorname{tr} g^{p^n} = 0$ for all $g \neq 1$. It follows that $0 = (k_1)^{p^n}$, and we conclude that if α is nilpotent, then $0 = k_1 = \operatorname{tr} \alpha$.

Finally, let I be a nil ideal of $K[G]$ and let $\gamma = \sum c_g g \in I$. Then $\gamma x^{-1} \in I$ is nilpotent for any $x \in G$, so the above yields $0 = \operatorname{tr} \gamma x^{-1} = c_x$. Thus $\gamma = 0$, and hence $I = 0$, as required. $\square$

Since any finitely generated field extension of $\mathrm{GF}(p)$ is separably generated, it is a simple matter to translate the argument of Theorem 2.5 to this context. In particular, Theorem 2.4 and Lemma 3.1 yield

Theorem 3.2. *Let K be a field of characteristic $p > 0$, write $K_0 = \mathrm{GF}(p)$, and let G be a p'-group.*

i. *If K/K_0 is not algebraic, then $K[G]$ is semiprimitive.*
ii. *If K/K_0 is algebraic, then $JK[G] = K \otimes_{K_0} JK_0[G]$.*

If G is a p'-group, then $K[G]$ is presumably always semiprimitive. But the converse is certainly not true; there are numerous groups G having elements of order p, but with $JK[G] = 0$. For example, we have

(1) $p = 2$ and $G = \langle x, y \mid y^{-1}xy = x^{-1}, y^2 = 1\rangle$ is infinite dihedral.
(2) $G = Z \wr Z_p$ is the wreath product of the infinite cyclic group Z by the cyclic group Z_p of order p.
(3) $G = Z_p \wr Z$ is again a wreath product and has a normal infinite elementary abelian p-subgroup.
(4) $G = \mathrm{Sym}_\infty$, the infinite locally finite symmetric group.

Note that (1) was the first such example; it appeared in [dW67]. Furthermore, the groups in (3) and (4) have primitive group algebras. The real answer to the semiprimitivity problem is most likely

Conjecture 3.3. *Let K be a field of characteristic $p > 0$ and let G be a group. Then $JK[G] \neq 0$ if and only if G has an element of order p "well placed" in G.*

Of course, before this can be proved, we must first determine what "well placed" means. To do this, it is necessary to compute numerous examples. However, we can get some idea of the possible meaning by considering a slightly different problem. For any ring R, let NR denote the join of all its nilpotent ideals. Thus NR is a characteristic nil ideal called the *nilpotent radical* of R. For general rings, it is neither nilpotent nor a radical, but we do have $NR \subseteq JR$.

Next, if A and B are subgroups of a group G, then the *finitary centralizer* of B in A is defined by

$$\mathbb{D}_A(B) = \{\, a \in A \mid |B : \mathbb{C}_B(a)| < \infty \,\}.$$

In other words, $\mathbb{D}_A(B)$ consists of all elements of A which almost centralize B; it is a subgroup of A normalized by $\mathbb{N}_G(A) \cap \mathbb{N}_G(B)$. Corresponding to this finitary centralizer is a finitary center, the *f.c.* or *finite conjugate center* of G, given by

$$\Delta(G) = \mathbb{D}_G(G) = \{\, x \in G \mid |G{:}\,\mathbb{C}_G(x)| < \infty \,\}.$$

Thus $\Delta(G)$ consists of all elements of G having only finitely many conjugates, and it is easy to see that $\Delta = \Delta(G)$ is a characteristic subgroup of G. Furthermore, we let $\Delta^+(G)$ be the set of torsion elements of Δ, that is the elements of finite order in the group. Surprisingly, $\Delta^+ = \Delta^+(G)$ is also a characteristic subgroup of G. Indeed, Δ/Δ^+ is a torsion free abelian group and Δ^+ is the join of all finite normal subgroups of G.

The following result is proved using a powerful coset counting argument known as the Δ-*method*.

Theorem 3.4. [dP62a,dP70a] *Let $\mathcal{W}(G)$ denote the set of finite normal subgroups of G, and let $\Delta^+ = \Delta^+(G) = \langle W \mid W \in \mathcal{W}(G)\rangle$. If $\operatorname{char} K = p > 0$, then*

i. $NK[G] = JK[\Delta^+] \cdot K[G]$.
ii. $JK[\Delta^+] = \bigcup_{W\in\mathcal{W}(G)} JK[W]$.
iii. $NK[G] \neq 0$ *if and only if Δ^+ contains an element of order p and hence if and only if G has a finite normal subgroup of order divisible by p.*

Note that (i) asserts that $JK[\Delta^+]$ is contained in $NK[G]$ and generates it as a right ideal. Furthermore, (iii) is an immediate consequence of parts (i) and (ii), along with Theorem 1.1. Thus "well placed" for this radical means that the element of order p is contained in $\Delta^+(G)$ or equivalently in some finite normal subgroup of G. We close this section with a simple, but quite useful, observation.

Lemma 3.5. [oV58] *If H is a normal subgroup of G of finite index n, then*

$$JK[G]^n \subseteq JK[H] \cdot K[G] \subseteq JK[G].$$

Furthermore, if p does not divide n, then $JK[H] \cdot K[G] = JK[G]$.

In particular, if $JK[H] = 0$ in the above, then $JK[G]$ is nilpotent and Theorem 3.4 can come into play. With this observation, it is now a simple exercise to prove that $K[G]$ is semiprimitive when $G = Z \wr Z_p$ or when $p = 2$ and G is infinite dihedral.

§4. Solvable and Linear Groups

This brings us to the early 1970's; it was time to compute some examples. We looked for families of groups which were sufficiently diverse to give us meaningful answers, yet simple enough to be dealt with effectively. Two obvious candidates were the families of solvable groups and linear groups. As it turned out, the solvable case yielded the most information and required the more interesting techniques.

Therefore we begin our exposition with these groups. We will ignore some earlier special case considerations and just deal with the general problem.

First, recall that G is said to be an *f.c. group* if $G = \Delta(G)$, or equivalently if all conjugacy classes of G are finite. Next, let G be any group, let H be a subgroup of G, and let I be a nonzero ideal of $K[G]$. Then an *intersection theorem* is a result which guarantees that $I \cap K[H] \neq 0$ under suitable assumptions on H, G/H, or I. There are numerous results of this nature in the literature, and Zalesskiĭ proved a marvellous one for solvable groups. Specifically, we have

Theorem 4.1. [aZ73a] *If G is a solvable group, then G has a characteristic f.c. subgroup $\mathfrak{Z}(G)$ with the following property. If K is any field and if I is a nonzero ideal of $K[G]$, then $I \cap K[\mathfrak{Z}(G)] \neq 0$.*

This *Zalesskiĭ subgroup* $\mathfrak{Z}(G)$ is the f.c. center of a finitary analog of the Fitting subgroup of a finite solvable group. Of course, if G is solvable and if $JK[G] \neq 0$, then the preceding theorem implies that $JK[G] \cap K[\mathfrak{Z}(G)] \neq 0$. Thus, the next step in the solution of the semiprimitivity problem for these groups is to deal with this intersection. For this, we require an interesting general result which is a noncommutative analog of the argument used to prove Theorem 2.4(iii).

Lemma 4.2. [dW67] *Let G be an arbitrary group, let $H \triangleleft G$, and suppose that $\alpha \in JK[G] \cap K[H]$. If x is any element of G of infinite order modulo H, then there exists a positive integer n such that*

$$\alpha\alpha^{x}\alpha^{x^2}\cdots\alpha^{x^n} = 0.$$

Here, of course, $\alpha^y = y^{-1}\alpha y$ for any $y \in G$. Now, if x has infinite order modulo H, then so does x^s for any positive integer s. Thus, each such x gives rise to a family of equations, with varying s and varying $n = n(s)$. These *Wallace equations* are rather unwieldy in general. Nevertheless, we were able to obtain a quite useful conclusion from them provided H is a solvable f.c. group.

For any element $\beta = \sum b_g g \in K[G]$, let us write $\operatorname{supp}\beta = \{\, g \in G \mid b_g \neq 0 \,\}$. In particular, the *support* of β is a finite subset of G which is nonempty when $\beta \neq 0$. Furthermore, let $\operatorname{quot}\beta$ denote the set of *quotients* xy^{-1} with $x, y \in \operatorname{supp}\beta$, and for any prime p let p-$\operatorname{quot}\beta$ denote the set of nonidentity elements of $\operatorname{quot}\beta$ having order a power of p. Finally, if L is any subgroup of G, we write

$$\sqrt{L} = \{\, x \in G \mid x^n \in L \text{ for some } n \neq 0 \,\}.$$

Obviously, $\sqrt{L} \supseteq L$, but this *root set* need not be a subgroup of G.

Proposition 4.3. [dPH72] *Let G be an arbitrary group, let H be a normal solvable f.c. subgroup of G, and let $\alpha \in JK[G] \cap K[H]$. Then*

$$G = \bigcup_{x \in p\text{-}\mathrm{quot}\,\alpha} \sqrt{\mathbb{C}_G(x)}.$$

It remained to translate the latter set theoretic union into a more understandable condition. To start with, notice that $G = \sqrt{\langle 1\rangle}$ is equivalent to G being a periodic group, and therefore the preceding root set equation is related to the Burnside problem. Fortunately, the Burnside problem is quite simple to deal with when G is solvable, and paper [dP73a] handled this more general situation. Specifically, it showed that if $G = \bigcup_1^n \sqrt{L_i}$ is a finite union of root sets of subgroups and if G is finitely generated and solvable, then some L_i must have finite index in G. By combining all these ingredients, we obtained

Theorem 4.4. [dPH72,dP73a,aZ73a] *Let G be a solvable group and let K be a field of characteristic $p > 0$. Then $JK[G] \neq 0$ if and only if $\mathfrak{Z}(G)$ contains an element x of order p which has finitely many conjugates under the action of each finitely generated subgroup of G.*

Note that the latter condition on x is equivalent to the assertion that if $x \in H \subseteq G$ with H finitely generated, then $x \in \Delta^+(H)$. In particular, if G is a finitely generated group, then this condition reduces to the assumption that $x \in \Delta^+(G)$, and of course this is precisely equivalent to the nonvanishing of $NK[G]$. In fact, fairly soon afterwards, Zalesskiĭ built upon the preceding, added an additional intersection theorem of sorts, and proved

Theorem 4.5. [aZ74a] *If G is a finitely generated solvable group and K is a field of characteristic $p > 0$, then $JK[G] = NK[G]$.*

In particular, in the above situation, we not only know when $K[G]$ is semiprimitive, we actually know the complete structure of $JK[G]$ by applying Theorem 3.4. Most of these results have now been extended to groups which have a finite normal series with f.c. factor groups. But these generalizations offer nothing new in the way of ideas or techniques. Now let us move on to consider linear groups over a field F. Here there are actually three different problems according to whether $\operatorname{char} F = 0$, $\operatorname{char} F = p = \operatorname{char} K$, or $\operatorname{char} F = q > 0$ with $q \neq \operatorname{char} K$. The first two cases have been completely settled, but there is still some work to be done on the third. The $\operatorname{char} F = p$ problem was dealt with using a complicated trace argument along with the solution of another variant of the Burnside problem for linear groups. The answer is quite similar to that for solvable groups and requires that we first define a particular characteristic f.c. subgroup $\mathcal{L}(G)$. This is done in a fairly simple manner, so $\mathcal{L}(G)$ is by no means as interesting as $\mathfrak{Z}(G)$.

Theorem 4.6. [dP73b,dP73c] *Let G be an F-linear group and assume that* $\operatorname{char} F = p = \operatorname{char} K$. *Then $JK[G] \neq 0$ if and only if $\mathcal{L}(G)$ has an element of order p which has finitely many conjugates under the action of each finitely generated subgroup of G.*

Now let us assume that G is a finitely generated F-linear group. If $\operatorname{char} F \neq p$, then it follows quite easily that G has a normal subgroup H of finite index which is residually a finite q-group for some prime $q \neq p$. Consequently, $JK[H] = 0$ and Lemma 3.5 implies that $JK[G]$ is nilpotent. On the other hand, if $\operatorname{char} F = p$, then it follows from the preceding theorem that $JK[G]$ is at least locally nilpotent. In other words, we have

Corollary 4.7. [dP74a] *If G is a finitely generated linear group and $\operatorname{char} K = p > 0$, then $JK[G] = NK[G]$.*

Thus a pattern began to emerge and we were led to

Conjecture 4.8. *If G is any finitely generated group and if $\operatorname{char} K = p > 0$, then $JK[G] = NK[G]$.*

There was even some corroborating evidence which held for arbitrary groups. Recall that the nilpotent radical is not a radical in general. Indeed, there exists a finitely generated K-algebra A with $N(A/NA) \neq 0$. But this cannot happen for group rings of finitely generated groups if the preceding conjecture is to hold. Fortunately, we were able to show

Theorem 4.9. [dP74a] *If G is any finitely generated group, then $K[G]$ is a finitely generated K-algebra satisfying $N\big(K[G]/NK[G]\big) = 0$. Consequently, if H is a subgroup of G of finite index, then $JK[H] = NK[H]$ if and only if $JK[G] = NK[G]$.*

We remark that this result, Theorem 4.5, and Corollary 4.7 were all proved using the following quite surprising radical-like property of the Δ^+ operator.

Lemma 4.10. [dP74a] *Let G be a finitely generated group and let H be a normal subgroup of G. If $H \subseteq \Delta^+(G)$, then $\Delta^+(G/H) = \Delta^+(G)/H$.*

It is easy to see that this lemma requires G to be finitely generated, and it does not hold for the Δ operator or indeed for the operator $\mathbb{Z}$, where $\mathbb{Z}(G)$ is the center of G. Unfortunately, this marks the extent of our knowledge of the semiprimitivity question for finitely generated groups. There has been no significant progress made on this problem since the above theorems appeared in 1973 and 1974.

§5. Locally Finite Groups

The obvious next step is to deal with arbitrary groups G under the assumption that we know the answer in the finitely generated case. For convenience, let $\mathcal{F}(G)$ denote the set of finitely generated subgroups of G. Then, motivated by Theorems 4.4 and 4.6, we define a local version of the f.c. center by

$$\Lambda(G) = \{\, x \in G \mid |H : \mathbb{C}_H(x)| < \infty \text{ for all } H \in \mathcal{F}(G)\,\}.$$

In other words,

$$\Lambda(G) = \bigcap_{H \in \mathcal{F}(G)} \mathbb{D}_G(H)$$

consists of all elements of G which have only finitely many conjugates under the action of each finitely generated subgroup of G. If we also let $\Lambda^+ = \Lambda^+(G)$ be the set of torsion elements of $\Lambda = \Lambda(G)$, then the known structure of Δ and Δ^+ translate to

Lemma 5.1. *Let G be an arbitrary group.*

i. Λ *and* Λ^+ *are characteristic subgroups of* G.
ii. Λ/Λ^+ *is torsion free abelian, and* Λ^+ *is a locally finite group.*
iii. *If* $H \triangleleft G$ *with* $H \subseteq \Lambda^+$, *then* $\Lambda^+(G/H) = \Lambda^+(G)/H$.

Of course, a group G is *locally finite* if every finitely generated subgroup is finite. For such groups, it follows easily that $\Lambda^+(G) = G$. Thus, the assertion of part (ii) that $\Lambda^+(G)$ is locally finite cannot be further sharpened. Notice also that part (iii) above asserts that the operator Λ^+ exhibits radical-like properties. This is clearly a local version of Lemma 4.10.

Now suppose $\alpha \in JK[G]$ and let H be any finitely generated subgroup of G with $\operatorname{supp}\alpha \subseteq H$. Then $\alpha \in JK[G] \cap K[H] \subseteq JK[H]$ and hence, if we happen to know that $JK[H] = NK[H]$, then we can use the structure of $NK[H]$, as described in Theorem 3.4, to better understand α. Specifically, we obtain

Theorem 5.2. [dP74a] *Let G be an arbitrary group and let K be a field of characteristic $p > 0$. If $JK[H] = NK[H]$ for all $H \in \mathcal{F}(G)$, then*

$$JK[G] = JK[\Lambda^+(G)] \cdot K[G].$$

In particular, it follows from Theorem 4.5 and Corollary 4.7 that if G is either locally solvable or locally linear, then $JK[G] = JK[\Lambda^+(G)] \cdot K[G]$. This is, in fact, how the semiprimitivity problem for characteristic 0 linear groups was settled. Namely, if G is such a group, then $JK[G]$ is generated by $JK[\Lambda^+(G)]$, and $\Lambda^+(G)$ is a locally finite characteristic 0 linear group. Thus $\Lambda^+(G)$ is abelian-by-finite and, with this, we can easily obtain a result quite similar to Theorem 4.6.

Notice also that if Conjecture 4.8 holds, then Theorem 5.2 reduces the semiprimitivity problem to the case of locally finite groups. In other words, the general problem has now been split into two parts. First, we must study the finitely generated case and show that $JK[G] = NK[G]$ for such groups. Then we must settle the problem for locally finite groups. In particular, this means that the locally finite case is also of crucial importance, and the remainder of this survey will be devoted to a discussion of this situation.

To start with, let us take another look at Theorems 4.4 and 4.6 in the context of locally finite groups. In each case, we have a normal f.c. subgroup H of G and an element $x \in H$ of order p. Since H is generated by its finite normal subgroups, it follows that x is contained in such a subgroup M. Thus M is a finite subnormal subgroup of G of order divisible by p, and it appears that these finite subnormal subgroups may be the key to the solution. But inclusion in the Jacobson radical is a local property, as will be apparent below, so a local version of subnormality may be more appropriate.

Let G be a locally finite group and let A be a finite subgroup of G. We say that A is *locally subnormal* in G, and write $A \operatorname{lsn} G$, if A is subnormal in B for all finite subgroups B of G with $A \subseteq B$. For example, if G is locally nilpotent, then every finite subgroup is locally subnormal. Basic properties are as follows.

Lemma 5.3. *Let G be a locally finite group and let K be a field.*

i. *$JK[G]$ is a nil ideal.*
ii. *If $A \lhd\lhd G$, then $JK[A] \subseteq JK[G]$.*
iii. *If $A \operatorname{lsn} G$, then $JK[A] \subseteq JK[G]$.*

Proof. We sketch the argument. For part (i), let $\alpha \in JK[G]$ and choose a finite subgroup H of G which contains the support of α. Then $\alpha \in JK[G] \cap K[H] \subseteq JK[H]$ by Lemma 2.3(ii), and $JK[H]$ is nilpotent since H is finite. Thus α is nilpotent, and $JK[G]$ is indeed a nil ideal. For part (ii), it suffices to assume that $A \lhd G$, and to show that $JK[A] \cdot K[G]$ is a nil right ideal of $K[G]$. To this end, let $\gamma \in JK[A] \cdot K[G]$ and write $\gamma = \sum_1^n \alpha_i \beta_i$ with $\alpha_i \in JK[A]$ and $\beta_i \in K[G]$. Since G/A is locally finite, there exists a finite subgroup B/A of G/A with $\operatorname{supp} \beta_i \subseteq B$ for all i. Then, by Lemma 3.5, $\gamma = \sum_1^n \alpha_i \beta_i \in JK[A] \cdot K[B] \subseteq JK[B]$, and hence γ is nilpotent, as required. Part (iii) follows in a similar manner. $\square$

If K is a field of characteristic $p > 0$, and if P is a locally finite p-group, then it follows from part (iii) above that $JK[P]$ is the *augmentation ideal* of $K[P]$, namely the kernel of the natural homomorphism $K[P] \to K$ given by $P \mapsto 1$. In particular, if $P = \mathbb{O}_p(G)$ is the largest normal p-subgroup of G, then $JK[P] \cdot K[G]$ is the kernel of the natural homomorphism $K[G] \to K[G/P]$, and this kernel is contained in $JK[G]$ by (ii) above. In other words, we have

$$JK[G]/(JK[P] \cdot K[G]) \cong JK[G/P],$$

and obviously $\mathbb{O}_p(G/P) = \langle 1 \rangle$. Because of this, it usually suffices to assume that $\mathbb{O}_p(G) = \langle 1 \rangle$.

As we will see, if $\mathbb{O}_p(G) = \langle 1 \rangle$, then the differences between locally subnormal subgroups, finite subnormal subgroups, and finite subgroups of normal f.c. subgroups essentially disappear. Note that we are interested in the p-elements of such a finite subgroup A, and hence our real concern is with $\mathbb{O}^{p'}(A)$, the characteristic subgroup of A generated by its Sylow p-subgroups. In other words, we can usually assume that $A = \mathbb{O}^{p'}(A)$. In the following definition, $\operatorname{len} A$ denotes the *composition length* of A, namely the common length of all composition series for A. Since A is finite, $\operatorname{len} A$ is certainly finite.

Now for any locally finite group G and fixed prime p, let $\int^p(G)$ be the characteristic subgroup of G generated by all $A \operatorname{lsn} G$ with $A = \mathbb{O}^{p'}(A)$. Furthermore, for each integer $n \geq 1$, let $\int_n^p(G)$ be the subgroup of G generated by all A lsn G with $A = \mathbb{O}^{p'}(A)$ and $\operatorname{len} A \leq n$. Then we have

Theorem 5.4. [dP74b] *Let G be a locally finite group with $\mathbb{O}_p(G) = \langle 1 \rangle$. Then $\int^p(G)$ is an ascending union of the characteristic f.c. subgroups $\int_n^p(G)$.*

Suppose, in the above situation, that A lsn G, $A = \mathbb{O}^{p'}(A)$, and say $\operatorname{len} A = n$. Then $A \subseteq \int_n^p(G)$, and the latter is a normal f.c. subgroup of G. In particular, since $\int_n^p(G)$ is generated by its finite normal subgroups, there exists such a subgroup B with $A \subseteq B \triangleleft \int_n^p(G)$. But $|B| < \infty$, so $A \triangleleft\triangleleft B$ and therefore $A \triangleleft\triangleleft G$. Furthermore, if we take B to be the normal closure of A in $\int_n^p(G)$, then $B = \mathbb{O}^{p'}(B)$ and $B \triangleleft\triangleleft G$ with subnormal depth at most 2. Thus these several concepts all merge into one.

To handle groups having normal p-subgroups, it is natural to define $\mathcal{S}^p(G) \supseteq \mathbb{O}_p(G)$ so that

$$\mathcal{S}^p(G)/\mathbb{O}_p(G) = \int^p(G/\mathbb{O}_p(G)).$$

Then $\mathcal{S}^p(G)$ is a characteristic subgroup of G with a fairly nice structure which can be read off from the preceding theorem. Furthermore, we have

Lemma 5.5. *Let $\operatorname{char} K = p$, and write $S = \mathcal{S}^p(G)$ and $P = \mathbb{O}_p(G)$.*

i. $JK[S] \cdot K[G] \subseteq JK[G]$.
ii. $JK[S]/(JK[P] \cdot K[S]) = JK[\int^p(G/P)] = \bigcup JK[A]$, *where the union of over all A lsn G/P with $A = \mathbb{O}^{p'}(A)$.*
iii. $JK[S] \neq 0$ *if and only if $S \neq \langle 1 \rangle$, or equivalently if and only if G has a locally subnormal subgroup of order divisible by p.*

For a number of reasons it appears that the set theoretic inclusion in (i) above may always be an equality. For example, it holds when G is a locally finite solvable group or an F-linear group with $\operatorname{char} F = 0$ or p. As we indicated earlier, the case of locally finite linear groups in characteristic $q \neq p$ has yet to be settled. Thus, we are led to the following

Conjecture 5.6. *If G is a locally finite group and K is a field of characteristic $p > 0$, then*

$$JK[G] = JK[\mathcal{S}^p(G)] \cdot K[G].$$

We will discuss additional corroborating evidence for this in the next section.

§6. Locally Solvable Groups

Before we proceed further, it is worthwhile to see what the latter two conjectures say about the semiprimitivity problem for group rings of arbitrary groups. To this end, let G be any group and let K be a field of characteristic $p > 0$. If H is a finitely generated subgroup of G, then according to Conjecture 4.8, $JK[H] = NK[H]$, and therefore Theorem 5.2 yields $JK[G] = JK[\Lambda^+(G)] \cdot K[G]$. But $\Lambda^+(G)$ is locally finite, so Conjecture 5.6 implies that $JK[\Lambda^+(G)] = JK[\mathcal{S}^p(\Lambda^+(G))] \cdot K[\Lambda^+(G)]$, and hence we have

$$JK[G] = JK[\mathcal{S}^p(\Lambda^+(G))] \cdot K[G].$$

Furthermore, Lemma 5.5 contains an appropriate description of $JK[\mathcal{S}^p(\Lambda^+(G))]$. In particular, it follows from the above and Lemma 5.5(iii) that $JK[G] \neq 0$ if and only if $\mathcal{S}^p(\Lambda^+(G)) \neq \langle 1 \rangle$, and hence if and only if G has an element x of order p contained in a locally subnormal subgroup of $\Lambda^+(G)$. With this, we now know what "well placed" means in Conjecture 3.3.

Of course, neither Conjecture 4.8 nor 5.6 has been proved, and we seem to be quite far from the general solution. Nevertheless, there has been significant progress made on the case of locally finite groups, so we return to this situation now. Indeed, for the remainder of this survey, G will always denote a locally finite group and K will be a field of characteristic $p > 0$. As we remarked, Conjecture 5.6 was shown, in [dP75a], to hold for solvable groups and F-linear groups with $\operatorname{char} F = 0$ or p. Furthermore, we have

Theorem 6.1. [dP75a] *Let G be a locally finite group.*

i. *$JK[\mathcal{S}^p(G)] \cdot K[G]$ is a semiprime ideal of $K[G]$. It is a prime ideal when $\Delta^+(G/\mathbb{O}_p(G)) = \langle 1 \rangle$.*
ii. *If H is a subgroup of finite index in G, then $JK[G] = JK[\mathcal{S}^p(G)] \cdot K[G]$ if and only if $JK[H] = JK[\mathcal{S}^p(H)] \cdot K[H]$.*

Of course, an ideal I of a ring R is said to be *semiprime* if $N(R/I) = 0$, and JR must necessarily have this property. Thus the above result at least partially corroborates Conjecture 5.6. We remark that Theorem 6.1 was surprisingly difficult to prove. It required intersection theorems from [DZ75], and a significant amount of group theory. Specifically, *a generalized Fitting subgroup* $\mathbb{F}^*(G)$ was defined and shown to have the following minimax property.

Theorem 6.2. [dP75a] *Let $G = \int^p(G)$ with $\mathbb{O}_p(G) = \langle 1 \rangle$, and set $F = \mathbb{F}^*(G)$.*

i. *$G = \mathbb{D}_G(F) = \{ g \in G \mid |F : \mathbb{C}_F(g)| < \infty \}$, and hence F is a characteristic f.c. subgroup of G.*
ii. *Suppose $G \triangleleft GB$ where B is a finite group with $|F : \mathbb{C}_F(B)| < \infty$. Then GB is generated by its locally subnormal subgroups.*

In other words, part (i) shows that F is small enough to be almost central in G, while part (ii) implies that it is big enough to control certain types of automorphisms of G. Next, we state and prove the elementary, but extemely useful

Lemma 6.3. [dP79a] *Let $H \triangleleft G$ with $JK[H] = NK[H]$. If $D = \mathbb{D}_G(H)$, then*

$$JK[G] = JK[D] \cdot K[G].$$

Proof. Since $D \triangleleft G$, Lemma 5.3(ii) implies that $JK[D] \cdot K[G] \subseteq JK[G]$. For the reverse inclusion, let $\alpha \in JK[G]$ and choose any subgroup $B \supseteq H$ with $|B/H| < \infty$ and $\alpha \in K[B]$. Then $\alpha \in JK[G] \cap K[B] \subseteq JK[B]$, and $JK[B] = NK[B] = JK[\Delta^+(B)] \cdot K[B]$ by Theorems 4.9 and 3.4(i). But $|B : H| < \infty$ and B is periodic, so $\Delta^+(B) = \mathbb{D}_B(H) = D \cap B$, and hence $\alpha \in JK[D \cap B] \cdot K[B] \subseteq JK[D \cap B] \cdot K[G]$. Since this holds for all such B, it follows easily that $\alpha \in JK[D] \cdot K[G]$. $\square$

The final result of this section deals with locally p-solvable groups. Its proof uses Δ-methods applied to finite subgroups of G, and makes crucial use of Theorem 6.1(ii) and the preceding result applied to $H = \mathbb{O}_{p'}(G)$. In addition, it requires a number of preliminary observations on finitary linear groups.

Theorem 6.4. [dP79a] *If G is a locally finite, locally p-solvable group and if K is a field of characteristic $p > 0$, then*

$$JK[G] = JK[\mathcal{S}^p(G)] \cdot K[G].$$

With this result, proved in 1979, we completed an intensive ten year attack on the semiprimitivity problem in characteristic $p > 0$. At this point, it seemed appropriate to move on to other tasks. The general locally finite case would surely require a better understanding of the finite simple groups, and the classification was not to be completed for several more years. But before we leave the 1970s, we should mention two special cases of Conjecture 5.6 which would serve as later test problems. To start with, if G is infinite simple, then it follows easily that $\mathcal{S}^p(G) = \langle 1 \rangle$. Furthermore, if $|G|_p < \infty$, that is if there is a bound on the orders of the finite p-subgroups of G, then $\mathcal{S}^p(G)$ is a finite normal subgroup of G. Thus we are led to

Conjecture 6.5. *Let G be a locally finite group.*

i. *If G is infinite simple group, then $JK[G] = 0$.*
ii. *If $|G|_p < \infty$, then $JK[G]$ is nilpotent.*

These were not considered at all during the decade of the 1980s, but they were solved in the affirmative quite recently using the known structure of infinite simple groups. It turned out that the wait was necessary.

§7. Infinite Simple Groups

Finally, we can discuss some recent progress on semiprimitivity. Again we assume that G is a locally finite group and that K is a field of characteristic $p > 0$. If π is any set of primes, we say that g is a π-*element* if $|g|$, the order of g, has all its prime factors in π. For convenience, we let G_π denote the set of π-elements of G, so that $1 \in G_\pi$ for all π. If X is a finite subset of $G^\# = G \setminus 1$, we say that $z \in G$ is a π-*insulator* of X if $z \in G_\pi$ and $zX \cap G_\pi = \emptyset$. Furthermore, we say that G is

π-insulated if every finite subset of $G^{\#}$ has a π-insulator. Note that, if $\pi = \{ p \}$ consists of the single prime p, then we use *p-element* and *p-insulated* instead of the more cumbersome $\{ p \}$-element and $\{ p \}$-insulated. The following is proved by a simple trace argument.

Lemma 7.1. *Let π be a set of primes containing $p = \operatorname{char} K$. If G is π-insulated, then $K[G]$ is semiprimitive.*

Surprisingly, this is all the group ring theory we need to settle Conjecture 6.5(i). The remainder of the long argument is entirely group theoretic in nature and requires a close look at the structure of locally finite simple groups. For our purposes, it suffices to assume that all such groups are countably infinite.

Suppose, for example, that $G = \mathrm{Alt}_\infty$ is the alternating group on the set of positive integers. If Alt_n denotes the subgroup of G moving points in $\{ 1, 2, \ldots , n \}$ and fixing the rest, then G is the ascending union of the groups Alt_n with $n \geq 5$, and hence G is an ascending union of finite simple groups. Unfortunately, this property is not always true. More typical is the case where G is the finitary special linear group $\mathrm{FSL}_\infty(F)$ with F a finite field. Here G consists of all countably infinite square F-matrices

$$g = \begin{bmatrix} \bar{g} & 0 \\ 0 & I \end{bmatrix}$$

where $\bar{g} \in \mathrm{SL}_n(F)$ for some n and I is the identity matrix on the remaining rows and columns. Notice that $\mathrm{FSL}_\infty(F)$ contains no nonidentity scalar matrix, so there is no need to form the projective group. Now it is clear that G is the ascending union of the finite subgroups $G_n \cong \mathrm{SL}_n(F)$ with $n \geq 4$, but this time the groups G_n are not simple. Instead, G_n has a normal subgroup M_n, corresponding to the scalar matrices, and $G_n/M_n \cong \mathrm{PSL}_n(F)$ is simple. Furthermore, the combined map

$$G_{n-1} \to G_n \to G_n/M_n \cong \mathrm{PSL}_n(F)$$

is easily seen to be an embedding. This is a precursor of the following fundamental result.

Lemma 7.2. [oK67] *Let G be a locally finite, countably infinite simple group. Then G has finite subgroups G_i for $i = 0, 1, 2, \ldots$ satisfying*

i. *$G_i \subseteq G_{i+1}$ and $G = \bigcup_0^\infty G_i$,*
ii. *$M_i \triangleleft G_i$ with $G_i/M_i = S_i$ a nontrivial simple group, and*
iii. *for all $i < j$, the composite map*

$$G_i \to G_j \to G_j/M_j = S_j$$

is an embedding.

In the above situation, we say that G is a *limit of the approximating sequence* $S_0, S_1, \dots$ and we write $G = \lim_{i\to\infty} S_i$. Of course, G is not uniquely determined by the simple groups S_i, but the approximating sequence does encode a surprising amount of information on the structure of G. To start with, the Classification Theorem (see [dG82]) asserts that the collection of finite simple groups is divided into finitely many infinite families and finitely many exceptions, the *sporadic groups*. Thus, since any subsequence of the triples (G_i, M_i, S_i) also determines an approximating sequence for G, we can assume that all S_i belong to the same infinite family. Now most of these families have a prime power parameter and all have an integer parameter n. Furthermore, it turns out that G is a linear group if and only if the parameter n is bounded throughout the sequence. The nonlinear case was settled first.

Theorem 7.3. [dPZ93] *Let G be a locally finite simple group which is not a linear group. Then G is p-insulated for any prime p. In particular, every group algebra $K[G]$ is semiprimitive.*

One aspect of the proof of this result deals with the maps $G_i \to G_j \to S_j$ which are by no means the obvious inclusions. Fortunately, this difficulty can be overcome with a simple idea implemented in a fairly tedious manner. The more interesting aspect of the argument really concerns the infinite groups Alt_∞, $\mathrm{FSL}_\infty(F)$, $\mathrm{FSU}_\infty(F)$, $\mathrm{FSp}_\infty(F)$, and $\mathrm{F\Omega}_\infty(F)$ where F is a finite field. Note that the latter four groups correspond to the families of Lie type for which the integer parameter n can become unbounded. The first group had actually been considered in 1972, and we sketch the clever proof.

Lemma 7.4. [eF72a] *If $G = \mathrm{Alt}_\infty$ or Sym_∞, then G is p-insulated for any prime p.*

Proof. If X is a finite subset of $G^\#$, then we can choose an even integer k so that the elements of $X \subseteq \mathrm{Sym}_\infty$ move only points in the set $\{1, 2, \dots, k\}$. Now define

$$z = (1 * \dots *)(2 * \dots *) \dots (k * \dots *)$$

where the $*$'s denote distinct points in $\{k+1, k+2, \dots\}$ and where $(j * \dots *)$ is a cycle of length p^j. Clearly $z \in \mathrm{Sym}_\infty$ is a p-element, and hence $g \in \mathrm{Alt}_\infty$ if p is odd. On the other hand, if $p = 2$, then z is the product of an even number of odd cycles, so again $z \in \mathrm{Alt}_\infty \subseteq G$.

Finally, let $x \in X$, and write x as a product of disjoint cycles which, by assumption, involve only the first k points. If $(j_1 j_2 \dots j_r)$ is such a nontrivial cycle in x, then zx (acting on the right) contains the cycle

$$(j_1 * \dots * j_2 * \dots * \dots j_r * \dots *)$$

which is the juxtaposition of the corresponding cycles in z. Since the j_i are distinct, the latter displayed cycle has length $p^{j_1} + p^{j_2} + \cdots + p^{j_r}$ and this is not a power of p. Thus zx is not a p-element, so $zX \cap G_p = \emptyset$ and G is p-insulated. $\square$

The corresponding proof for the infinite size matrix groups is much more complicated. In some sense, these groups divide naturally into the four cases

Case 1:	$G = \mathrm{FSL}_\infty(F)$	$\operatorname{char} F \neq p$
Case 2:	$G = \mathrm{FSU}_\infty(F)$, $\mathrm{FSp}_\infty(F)$, $\mathrm{F\Omega}_\infty(F)$	$\operatorname{char} F \neq p$
Case 3:	$G = \mathrm{FSL}_\infty(F)$	$\operatorname{char} F = p$
Case 4:	$G = \mathrm{FSU}_\infty(F)$, $\mathrm{FSp}_\infty(F)$, $\mathrm{F\Omega}_\infty(F)$	$\operatorname{char} F = p$

and these are dealt with in turn. The difficulty increases as we go down the list and reaches a crescendo when we hit the bottom.

Now on to the simple linear groups. Here, we have a wonderful characterization of such groups based on the Classification Theorem.

Theorem 7.5. [HS84,sT83] *Let G be a locally finite simple group. If G is an infinite linear group, then G is a group of Lie type over a locally finite field F.*

Of course, the field F is *locally finite* if $\operatorname{char} F = q > 0$ and F is contained between $\mathrm{GF}(q)$ and its algebraic closure. It follows from the above characterization that G contains a 1-parameter family of q-elements and, using this and the Zariski topology on G, we obtain

Theorem 7.6. [dP94a] *Let G be a locally finite simple group. If G is an infinite linear group over a locally finite field F of characteristic $q > 0$, then G is $\{p, q\}$-insulated for any prime p. In particular, every group algebra $K[G]$ is semiprimitive.*

Thus Theorems 7.3 and 7.6 settle Conjecture 6.5(i) in the affirmative. Furthermore, with a little more work and a knowledge of the Schur multipliers of the groups of Lie type, we can prove that if G is infinite simple, then any twisted group algebra $K^t[G]$ is semiprimitive. This is not merely of academic interest; the twisted result is actually required in the next section.

§8. Groups with Bounded p-Part

Finally, we consider Conjecture 6.5(ii). Again G is a locally finite group, and we let $|G|_p$ denote the supremum of the orders of its finite p-subgroups. In view of the Sylow theorems, it is clear that $|G|_p < \infty$ if and only if G satisfies the ascending chain condition on finite p-subgroups and hence if and only if G has no infinite p-subgroup. Of course, $|G|_p = 1$ is equivalent to G being a p'-group. Now if $|G|_p < \infty$, then we have a finite parameter to induct on, and therefore Lemma 7.2 and Theorem 7.5 yield

Lemma 8.1. *Let G be a locally finite group with $|G|_p < \infty$. Then G has a finite subnormal series*

$$\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with each quotient G_i/G_{i-1} either

i. *a p'-group, or*
ii. *a finite simple group, or*
iii. *an infinite simple group of Lie type.*

In other words, G has a finite subnormal series with factors for which we know the solution to the semiprimitivity problem. This is, of course, true now. But, at the time this work was going on, Theorem 7.6 had not yet been proved. Thus, it was necessary to use an earlier special case of that result, from [aZ92a], to obtain

Theorem 8.2. [dP93a] *Let G be a locally finite group with $|G|_p < \infty$. If* $\operatorname{char} K = p > 0$*, then $JK[G]$ is nilpotent.*

The proof of this result starts with a simple reduction which allows us to assume that G has no finite normal subgroup of order divisible by p, and we are left with the task of showing that $JK[G] = 0$. Some aspects of the latter semiprimitivity argument will be discussed in the more general context of

Theorem 8.3. [dP95a] *Let $K[G]$ be the group algebra of a locally finite group G over a field K of characteristic $p > 0$. Suppose that G has a finite subnormal series*

$$\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with each quotient G_i/G_{i-1} either

i. *a p'-group, or*
ii. *a nonabelian simple group, or*
iii. *generated by its locally subnormal subgroups.*

Then $K[G]$ is semiprimitive if and only if G has no locally subnormal subgroup of order divisible by p.

A brief outline of the proof of semiprimitivity here is as follows. First, we can assume that K is algebraically closed. Then we proceed by induction on the number of factors of the subnormal series for G which are infinite simple but not p'-groups. It turns out that, using Lemma 6.3, we can quickly reduce to the case of just one such factor. Indeed, it suffices to assume that G has a normal subgroup N with $G/N = H$ an infinite simple group containing an element of order p, and with $|N : \mathbb{C}_N(g)| < \infty$ for all $g \in G$. Furthermore, N is a p'-group and we can suppose that G has no nontrivial f.c. homomorphic image. In other words, the pair (G, N) is what we call a *p'-f.c. cover* of H. Now if N is central in G, then G is a central cover of H and $K[G]$ is a subdirect product of various twisted group algebras $K^t[H]$. Thus, the twisted analogs of Theorems 7.3 and 7.6 apply here and yield the result.

On the other hand, if N is not central in G, then we show that $H = G/N$ is a finitary linear group over the Galois field $\mathrm{GF}(q)$ for some a prime q involved in the subgroup N. It follows that H cannot be a linear group, so the results of [jH88], [jH95a] and [jH95b] imply that H is isomorphic to one of the stable finitary groups Alt_∞, $\mathrm{FSL}_\infty(F)$, $\mathrm{FSU}_\infty(F)$, $\mathrm{FSp}_\infty(F)$, or $\mathrm{F}\Omega_\infty(F)$ for some locally finite field F of characteristic q.

Finally, we define a stronger version of p-insulation and we show that if H is *strongly p-insulated*, then any p'-f.c. cover G of H is p-insulated and hence satisfies $JK[G] = 0$. Thus all that remains is to prove that the stable groups H, as listed above, are strongly p-insulated. Indeed, since N is a p'-group, we have $q \neq p$, and thus we need only consider the stable groups in characteristic $q \neq p$. This turns out to be a great simplification but, for the sake of completeness, paper [dP95a] shows that the stable groups in characteristic p are also strongly p-insulated. The latter fact requires a rather long and unpleasant argument.

This brings us to the present, and obviously there is still much to be done on the semiprimitivity problem. Fortunately, there is good reason to believe that the locally finite case can be settled in the not too distant future. For example, a close look at the proof of Theorem 6.4 shows that it actually contains an outline for a general argument. But there are still fundamental problems, mostly concerning finite groups, which have to be resolved before this task can be completed.

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Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706
E-mail address: passman@math.wisc.edu