

Notes on Planetary Motion

(1) The motion is planar

Use 3-dimensional coordinates with the sun at the origin. Since $F = mA$ and the gravitational pull is in towards the sun, the acceleration A is parallel to the position vector R . Now

$$\frac{d(R \times V)}{dt} = \frac{dR}{dt} \times V + R \times \frac{dV}{dt} = V \times V + R \times A.$$

But $V \times V = 0$ and $R \times A = 0$, since R and A are parallel vectors. Thus $d(R \times V)/dt = 0$, so $R \times V$ is a constant, say \tilde{C} . Since $R \perp (R \times V)$, we see that the position vector R is in the plane containing the origin O and perpendicular to \tilde{C} . Note that we only used the fact that F is parallel to R here (a central force field). We did not use the magnitude of F .

(2) Polar coordinates in the plane

Assume that the motion is in the (x, y) -plane and use polar coordinates (r, θ) . Thus, both r and θ are functions of t . Define the unit vectors

$$U_r = i \cos \theta + j \sin \theta, \quad U_\theta = -i \sin \theta + j \cos \theta.$$

Note that $U_r \perp U_\theta$, $dU_r/d\theta = U_\theta$ and $dU_\theta/d\theta = -U_r$, so

$$\frac{dU_r}{dt} = \frac{d\theta}{dt} U_\theta, \quad \frac{dU_\theta}{dt} = -\frac{d\theta}{dt} U_r.$$

Since $x = r \cos \theta$ and $y = r \sin \theta$, we have $R = xi + yj = rU_r$ and hence

$$V = \frac{dR}{dt} = \frac{d(rU_r)}{dt} = \frac{dr}{dt} U_r + r \frac{dU_r}{dt} = \frac{dr}{dt} U_r + r \frac{d\theta}{dt} U_\theta.$$

Differentiating again, we get

$$A = \frac{dV}{dt} = \left[\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] U_r + \left[r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right] U_\theta.$$

Observe that the U_θ component of A is equal to

$$r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} = \frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right).$$

(3) Central force fields and Kepler's second law

Assume that the force F is directed in or out from the origin O . This is a central force field. As in part (1), we know that the motion is planar. Use polar coordinates. Since $F = mA$, the acceleration A is in the same direction as R and hence the same direction as U_r . Thus the U_θ -component of A is zero. This yields

$$\frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) = 0,$$

so $r^2(d\theta/dt) = h$, a constant.

Recall that the area $\mathcal{A}(t)$ swept out by the moving position vector R is

$$\mathcal{A}(t) = \int_0^t \frac{1}{2} r^2 d\theta,$$

so

$$\frac{d\mathcal{A}}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} h$$

and $\mathcal{A} = (ht)/2$. This is Kepler's second law of motion which says "The radius vector from the sun to a planet sweeps out area at a constant rate."

(4) Initial conditions

Assume that R_0 , the initial position vector of a planet (at time $t = 0$) is in the positive x -direction, so that $\theta = 0$, and that it represents the closest approach of the planet to the sun. Then $dr/dt = 0$ at this point, so $V = (dr/dt)U_r + r(d\theta/dt)U_\theta$ implies that V_0 is in the U_θ direction. But $U_r = i$ and $U_\theta = j$ when $\theta = 0$, so $V_0 = v_0 j$, where v_0 is the initial speed. Since $v_0 = r(d\theta/dt)$ at $t = 0$, we get $r_0 v_0 = r^2(d\theta/dt) = h$.

(5) Conic sections and Kepler's first law

We now use Newton's law of gravitational attraction, namely $F = -(GMm/r^2)U_r$, where M is the mass of the sun, m is the mass of the planet, and G is the universal gravitational constant. Since $F = mA$, we get $A = -(\mu/r^2)U_r$, where $\mu = GM$. Furthermore, $r^2(d\theta/dt) = h$, so

$$\frac{dV}{dt} = A = -\frac{\mu}{r^2}U_r = -\frac{\mu}{h} \frac{d\theta}{dt} U_r = \frac{\mu}{h} \frac{dU_\theta}{dt}.$$

Taking antiderivatives of both sides, we get $V = (\mu/h)U_\theta + C$, where C is a constant. Now at $t = 0$, we have $V = v_0 j$ and $U_\theta = j$, so $C = (v_0 - \mu/h)j$ and

$$V = (\mu/h)U_\theta + (v_0 - \mu/h)j.$$

Next, we take the dot product of this expression with U_θ using the facts that $V \cdot U_\theta = r(d\theta/dt)$ (for a general V), $j \cdot U_\theta = \cos \theta$ and $U_\theta \cdot U_\theta = 1$. We get

$$r \frac{d\theta}{dt} = \frac{\mu}{h} + \left(v_0 - \frac{\mu}{h}\right) \cos \theta.$$

But $r(d\theta/dt) = h/r$, so

$$\frac{h}{r} = \frac{\mu}{h} + \left(v_0 - \frac{\mu}{h}\right) \cos \theta$$

and solving for r yields

$$r = \frac{h^2/\mu}{1 + [(v_0 h/\mu) - 1] \cos \theta}.$$

Since $r_0 v_0 = h$ and $\mu = GM$, we get

$$r = \frac{h^2/\mu}{1 + e \cos \theta}$$

where $e = (r_0 v_0^2 - GM)/(GM)$ and $h^2/\mu > 0$. We recognize the above as the equation of a conic section with one focus at the origin and with eccentricity equal to e . Note that $e \geq 0$ since r is minimal when $\theta = 0$, and hence $r_0 v_0^2 \geq GM$. This is Kepler's first law, namely "The orbit of each planet is an ellipse with the sun at one focus."

(6) Comets and comments

For more general objects like comets, there are additional options. If $r_0 v_0^2 = GM$, we get $e = 0$ and the orbit is a circle. If $r_0 v_0^2 < 2GM$, then $e < 1$ and the orbit is an ellipse. If $r_0 v_0^2 = 2GM$, then $e = 1$ and we have a parabola. Finally, if $r_0 v_0^2 > 2GM$, then the orbit is a hyperbola. Do there exist comets with circular, hyperbolic or parabolic orbits?

Remember that Isaac Newton invented calculus, introduced the basic laws of motion and gravitational force, and then used all of this machinery to derive Kepler's laws.

(7) Periods of revolution

Kepler's third law relates the period (year) of a planet to the length of its semimajor axis. To understand this, recall that if

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is an ellipse with semimajor axis a (so that $a \geq b$) and with eccentricity e , then we know that the area of the ellipse is equal to πab and that $b^2 = a^2 - c^2 = a^2 - (ae)^2 = a^2(1 - e^2)$.

Now suppose that the planet of interest has the above equation if the axes are suitably shifted. If τ is the period (year) of the planet, then the area formulas yield $\pi ab = h\tau/2$, so $h = 2\pi ab/\tau$ and

$$h^2 = \frac{4\pi^2 a^2 b^2}{\tau^2} = \frac{4\pi^2 a^4 (1 - e^2)}{\tau^2}.$$

Also, using the polar equation for r , we know that $2a$ is equal to r at $\theta = 0$ plus r at $\theta = \pi$. Thus

$$2a = \frac{h^2/\mu}{1 + e} + \frac{h^2/\mu}{1 - e} = \frac{2h^2/\mu}{1 - e^2},$$

so

$$h^2 = a\mu(1 - e^2).$$

Setting the two formulas for h^2 equal to each other, canceling a factor of a and $1 - e^2$, we get $GM = \mu = 4\pi^2 a^3/\tau^2$, so

$$\frac{\tau^2}{a^3} = \frac{4\pi^2}{GM} = \gamma.$$

Note that the right hand term γ depends only on the mass M of the sun. Thus it is the same for all planets and we obtain Kepler's third law, namely

“The square of the period of revolution of a planet is proportional to the cube of the semimajor axis of its orbit.”

Exercises

1. A particle moves in 2-space with polar coordinates given by

$$r = 1 + \sin \theta, \quad \theta = 2t$$

Describe R , V and A in terms of t , U_r and U_θ .

2. A particle moves through 2-space under the action of a central force field (centered at the origin). If $\theta = e^{2t}$ and $r(0) = 5$, find r as a function of t and then as a function of θ .