

POLYCYCLIC RESTRICTED LIE ALGEBRAS

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Dedicated to the memory of Professor A. I. Kostrikin

ABSTRACT. We define a polycyclic restricted Lie algebra to be the Lie analog of a polycyclic group, and we describe the structure of poly(cyclic or finite-dimensional) restricted Lie algebras. In particular, we prove that these are precisely the restricted Lie algebras whose restricted enveloping algebras have polynomial growth.

1. POLYCYCLIC RESTRICTED LIE ALGEBRAS

Denote the ground field by K and let $\text{char } K = p > 0$. We refer the reader to [4] for the definition and basic properties of a restricted Lie algebra. Let L be such an algebra over the field K . If the K -subspace $H \subseteq L$ is closed under the Lie bracket, then H is said to be a *Lie subalgebra* of L . In this case, one defines its p -*hull* to be $H_p = \langle h^{p^n} \mid h \in H, n \geq 0 \rangle_K$. Properties of the p -mapping imply that H_p is the minimal restricted subalgebra containing H , and that $[H_p, H_p] = [H, H]$.

We say that a restricted Lie algebra L is *cyclic* if it is generated by one element e as a p -algebra, that is $L = H_p$ where $H = \langle e \rangle_K$. It follows from the above that L is abelian, and there are two possibilities. First, we say that e is *algebraic* with respect to the p -mapping if there exists an integer n such that $e^{p^n} = \alpha_{n-1}e^{p^{n-1}} + \cdots + \alpha_1e^p + \alpha_0e$, with $\alpha_i \in K$. In this case, L is finite dimensional. On the other hand, if all p -powers of e are linearly independent, then e is *transcendental* and L is infinite dimensional.

Let $Z = \langle z_1, \dots, z_s \rangle_p$ be an abelian p -algebra, and suppose that all elements $\{z_i^{p^n} \mid 1 \leq i \leq s, n \geq 0\}$ are linearly independent. Then we refer to Z as the *free abelian* restricted Lie algebra of rank s .

The notion of a polycyclic group is well known [7]. By analogy, we say that a restricted Lie algebra L is *polycyclic* if there exists a series

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of restricted subalgebras

$$(1) \quad L = L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots \supseteq L_m = 0$$

with $L_{i-1} \triangleright L_i$ and L_{i-1}/L_i cyclic for $i = 1, \dots, m$. The algebra is said to be *almost polycyclic* if it has a polycyclic restricted subalgebra of finite codimension. Finally, L is called *poly(cyclic or finite-dimensional)* if it has a series (1) such that all factors L_{i-1}/L_i are either cyclic or finite dimensional. We also refer to these as *poly-CF* algebras.

Lemma 1. *Let L be a poly-CF restricted Lie algebra. Then $L = H_p$ for some finite-dimensional Lie subalgebra $H \subseteq L$.*

Proof. We prove the statement by induction on the length of the poly-CF chain (1). We consider the base of induction, namely $m = 1$. Here, if L is finite dimensional, then there is nothing to prove. On the other hand, if $L = \langle e \rangle_p$, then we take $H = \langle e \rangle_K$.

Now suppose that $L \triangleright L_1 \neq 0$. By the inductive assumption, we have $L_1 = Q_p$, where Q is a finite-dimensional Lie subalgebra of L_1 with K -basis $\{z_1, \dots, z_s\}$. Assume first that $\dim L/L_1 < \infty$, and let $v_1, \dots, v_a \in L$ with $L/L_1 = \langle \bar{v}_1, \dots, \bar{v}_a \rangle_K$, where $\bar{\cdot} : L \rightarrow L/L_1$. Define the finite-dimensional subspace H of L by

$$H = \langle v_i, [v_i, v_j], [v_i, Q], Q \mid 1 \leq i, j \leq a \rangle_K.$$

We claim that $[L, L] \subseteq H$. Indeed, an arbitrary element $x \in L$ can be written as

$$(2) \quad x = \sum_{i=1}^a \alpha_i v_i + \sum_{j=1}^s \sum_{n=0}^{\infty} \lambda_{j,n} z_j^{p^n}, \quad \alpha_i, \lambda_{j,n} \in K.$$

and observe that for $n > 0$ we have

$$(3) \quad [v_i, z_j^{p^n}] = [\dots [v_i, \underbrace{z_j, \dots, z_j}_{p^n \text{ times}}] \dots] \in [L_1, L_1] \subseteq Q \subseteq H.$$

Thus $[v_i, L] \subseteq H$ and, since $L = L_1 + \langle v_1, \dots, v_a \rangle_K$ and $[L_1, L_1] \subseteq H$, it follows that $[L, L] \subseteq H$. In particular, H is a Lie subalgebra of L , and (2) implies that $L = H_p$.

Finally, suppose $L/L_1 = \langle \bar{v}, \bar{v}^p, \dots, \bar{v}^{p^n}, \dots \rangle_K$ is cyclic, with $v \in L$. Then any arbitrary element $x \in L$ can be written as

$$(4) \quad x = \sum_{k=0}^{\infty} \alpha_k v^{p^k} + \sum_{j=1}^s \sum_{n=0}^{\infty} \lambda_{j,n} z_j^{p^n}, \quad \alpha_k, \lambda_{j,n} \in K,$$

and here we set $H = \langle v, [v, Q], Q \rangle_K$. Then using (3) with $v = v_i$, we see that $[v, L] \subseteq Q + [v, Q] \subseteq H$. Furthermore, for $k \geq 0$, we have $[v^{p^k}, L] \subseteq [v, \dots, [v, L] \dots] \subseteq [v, L] \subseteq H$. In particular, since

$[L_1, L_1] \subseteq Q \subseteq H$, we conclude that $[L, L] \subseteq H$. Thus H is a Lie subalgebra of L , and (4) implies that $L = H_p$. \square

2. GROWTH OF p -ALGEBRAS AND ENVELOPING ALGEBRAS

Let A be an associative (or Lie) algebra generated by a finite set X . We denote by $A^{(X,n)}$ the subspace of A spanned by all monomials in X of length not exceeding n (including the identity in the case of associative algebras). On the other hand, if A is a restricted Lie algebra, then we define $A^{(X,n)} = \langle [x_1, \dots, x_s]^{p^k} \mid x_i \in X, sp^k \leq n \rangle_K$. In either situation, we consider the two *growth functions* and the *Hilbert-Poincaré series* defined by

$$\begin{aligned}\Omega_A(n) &= \Omega_A(X, n) = \dim_K A^{(X,n)} \\ \lambda_A(n) &= \Omega_A(n) - \Omega_A(n-1) \\ \mathcal{H}_X(A, t) &= \sum_{n=0}^{\infty} \lambda_A(n) t^n.\end{aligned}$$

The growth functions and the Hilbert-Poincaré series clearly depend on the choice of the generating set X . Furthermore, it is easy to see that exponential growth is the highest possible growth for Lie and associative algebras. The growth function $\Omega_A(n)$ is compared with the polynomial functions n^k , $k \in \mathbb{R}^+$ by computing the *upper and lower Gelfand-Kirillov dimensions* [5], namely

$$\begin{aligned}\text{GKdim } A &= \limsup_{n \rightarrow \infty} \frac{\ln \Omega_A(n)}{\ln n} \\ \underline{\text{GKdim}} A &= \liminf_{n \rightarrow \infty} \frac{\ln \Omega_A(n)}{\ln n}.\end{aligned}$$

Next, we introduce an additional invariant especially appropriate for the growth of restricted Lie algebras. Observe that the free abelian p -algebra $Z = \langle z_1^{p^s}, \dots, z_k^{p^s} \mid s \geq 0 \rangle_K$ is generated by $X = \{z_1, \dots, z_k\}$, and we have $Z^{(X,n)} = \langle z_i^{p^s} \mid p^s \leq n \rangle_K$, so $\Omega_Z(X, n) = k([\log_p n] + 1)$. Thus we define the L_p -dimension of a finitely generated restricted Lie algebra A to be

$$L_p \dim A = \lim_{n \rightarrow \infty} \frac{\Omega_A(X, n)}{\log_p n}.$$

This value does not depend on the choice of the generating set. Indeed, suppose that X' is another generating set for A , with $X' \subseteq A^{(X,m)}$.

Then $A^{(X',n)} \subseteq A^{(X,nm)}$ for all $n \in \mathbb{N}$, and hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\Omega_A(X', n)}{\log_p n} &\leq \lim_{n \rightarrow \infty} \frac{\Omega_A(X, nm)}{\log_p n} \\ &= \lim_{n' \rightarrow \infty} \frac{\Omega_A(X, n')}{\log_p n' + C} = \lim_{n \rightarrow \infty} \frac{\Omega_A(X, n)}{\log_p n}. \end{aligned}$$

The reverse inequality follows by symmetry. Note that in the above example, we have $L_p \dim Z = k$.

For finitely generated groups, the growth is defined in a similar way, and one easily observes that the growth of a group G coincides with the growth of the group ring $K[G]$ (see [5]). On the other hand, the growth increases when we transfer from a Lie algebra to its universal enveloping algebra. This situation is precisely described in [8] and [9].

Suppose that an algebra A is generated by a finite set X . Then the X -length of an element $x \in A$ is given by

$$l_A(x) = \min\{n \mid x \in A^{(X,n)}\}.$$

Now let L be a restricted Lie algebra and construct an ordered basis compatible with the filtration $L^{(X,1)} \subseteq L^{(X,2)} \subseteq \dots$. In other words, $\{w_1, \dots, w_{\Omega_L(1)}\}$ is a basis for $L^{(X,1)}$ and $\{w_{\Omega_L(n-1)+1}, \dots, w_{\Omega_L(n)}\}$ determines a basis of $L^{(X,n)}$ modulo $L^{(X,n-1)}$. By the Poincaré-Birkhoff-Witt theorem, the standard monomials

$$w_{i_1}^{\alpha_1} w_{i_2}^{\alpha_2} \cdots w_{i_r}^{\alpha_r}, \quad i_1 < i_2 < \cdots < i_r, \quad 0 \leq \alpha_j < p,$$

form a basis for the restricted enveloping algebra $u(L)$ (see [4]). Furthermore, by arguments similar to those of [6], one can prove that

$$l_{u(L)}(w_{i_1}^{\alpha_1} \cdots w_{i_r}^{\alpha_r}) = \alpha_1 l_L(w_{i_1}) + \cdots + \alpha_r l_L(w_{i_r}),$$

and that this basis is compatible with the filtration on $u(L)$. It follows that $\lambda_{u(L)}(n)$ coincides with the number of solutions of the Diophantine equation

$$\begin{aligned} 1(x_{1,1} + \cdots + x_{1,\lambda_L(1)}) + 2(x_{2,1} + \cdots + x_{2,\lambda_L(2)}) \\ + \cdots + n(x_{n,1} + \cdots + x_{n,\lambda_L(n)}) = n, \quad 0 \leq x_{i,j} < p. \end{aligned}$$

Indeed, if we let $b_n = \lambda_L(n)$ and $a_n = \lambda_{u(L)}(n)$, for all $n \in \mathbb{N}$, then this Diophantine equation leads to the formal power series relation [11]

$$(5) \quad \mathcal{H}_X(u(L), t) = \sum_{n=0}^{\infty} a_n t^n = \prod_{n=1}^{\infty} (1 + t^n + t^{2n} + \cdots + t^{(p-1)n})^{b_n}.$$

Now we apply some ideas from [10] and [2].

Lemma 2. *Let L be a finitely generated restricted Lie algebra and suppose that its restricted enveloping algebra $u(L)$ has polynomial growth. Then there exists a finite-dimensional Lie subalgebra H with $L = H_p$.*

Proof. Suppose that L is generated by the finite set X , and let H be the Lie subalgebra generated by X . Denote by $H^{(X,n)}$, $n \in \mathbb{N}$ the linear span of Lie monomials in X of length not exceeding n . If H is infinite dimensional, then all inclusions $H^{(X,n-1)} \subseteq H^{(X,n)}$, $n \in \mathbb{N}$ must be strict, and we can choose elements $v_n \in H^{(X,n)} \setminus H^{(X,n-1)}$, $n \in \mathbb{N}$. Let us fix n , set $m = \lfloor \sqrt{n} \rfloor - 1$, and consider all elements of type

$$(6) \quad v = v_1^{\beta_1} \cdots v_m^{\beta_m}, \quad \beta_i \in \{0, 1\}.$$

Since $l_{u(L)}(v) = \sum_{s=1}^m \beta_s s \leq m(m+1)/2 \leq n$, we see that all elements given in (6) are linearly independent and belong to $u(L)^{(X,n)}$. Thus $\Omega_{u(L)}(n) \geq 2^m \geq 2^{\sqrt{n}-2}$, contradicting the fact that $u(L)$ has polynomial growth. Hence $\dim_K H < \infty$, and the result follows from $L = H_p$. \square

This proof also yields the following gap in the growth of restricted enveloping algebras. We use the notation of [9].

Corollary 1. *If L is a finitely generated restricted Lie algebra, then $\text{Dim}^3 u(L) \notin (0, 1)$.*

3. THE MAIN RESULT

The celebrated theorem of Gromov states that a finitely generated group has polynomial growth if and only if it is nilpotent-by-finite [3]. Thus the equivalence (4 \Leftrightarrow 5) in the theorem below is the Lie analogue of Gromov's result.

Theorem. *Suppose that L is a restricted Lie algebra over a field of characteristic p . The following properties are equivalent.*

- (1) L is almost polycyclic.
- (2) L is poly-CF.
- (3) $L = H_p$, where H is a finite-dimensional Lie subalgebra.
- (4) $L = R \oplus Z$, where R is a finite-dimensional Lie subalgebra and Z is a central free abelian p -algebra of finite rank.
- (5) L is finitely generated and $\text{GKdim } u(L) < \infty$.
- (6) L is finitely generated and $\text{L}_p \dim L < \infty$.

Proof. (1 \Rightarrow 2) follows from the well-known fact that any restricted subalgebra of L of finite codimension, contains a restricted ideal of L of finite codimension [1].

(2 \Rightarrow 3) is the conclusion of Lemma 1.

(3 \Rightarrow 4) Suppose that $L = H_p$, where $H = \langle v_1, \dots, v_a \rangle_K$ is a finite-dimensional Lie subalgebra. Since H is a finite-dimensional Lie ideal of L , the operators $(\text{ad } v_i)_H, (\text{ad } v_i^p)_H, (\text{ad } v_i^{p^2})_H, \dots$ are linearly dependent. Thus there exists an element $w_i \in L$ with

$$w_i = \alpha_{i,0}v_i + \alpha_{i,1}v_i^p + \cdots + \alpha_{i,d_i}v_i^{p^{d_i}}, \quad \alpha_{i,j} \in K,$$

$\alpha_{i,d_i} \neq 0$ and $(\text{ad } w_i)_H = 0$. But $L = H_p$, so $(\text{ad } w_i)_L = 0$ and hence w_i is central in L . Let V be the finite-dimensional subspace of L spanned by all $v_i^{p^j}$ with $i = 1, \dots, a$ and $0 \leq j < d_i$. Then $V \supseteq H$ and it is easy to see that

$$L = \langle v_1, \dots, v_a \rangle_p = \langle w_1, \dots, w_a \rangle_p + V.$$

For each $n \in \mathbb{N}$, define

$$W_n = \langle w_1^{p^n}, \dots, w_a^{p^n} \mid i \leq n \rangle_K + V$$

so that $W_0 \subseteq W_1 \subseteq \cdots$ is a filtration of L by finite-dimensional subspaces. Set $\overline{W}_n = W_n/W_{n-1}$, let $f(n) = \dim_K \overline{W}_n$, and note that $\overline{W}_{n+1} = \langle \overline{W}_n^p \rangle_K$ for $n \geq 1$. Thus $f(n) \geq f(n+1)$ and this decreasing sequence of integers eventually stabilizes, say $f(n) = k$ for all $n \geq N$. Without loss of generality, we may suppose that $w_1^{p^N}, \dots, w_k^{p^N}$ are linearly independent modulo W_{N-1} , and we set $z_i = w_i^{p^N}$ for $i = 1, \dots, k$. It is now easy to verify that $Z = \langle z_1, \dots, z_k \rangle_p$ is a central free abelian p -algebra and that $L = W_{N-1} \oplus Z$. Of course, $R = W_{N-1}$ is a finite-dimensional subalgebra of L since $[L, L] = [H, H] \subseteq H \subseteq R$.

(4 \Rightarrow 1) is clear, and therefore (1), (2), (3) and (4) are equivalent.

(5 \Rightarrow 3) follows from Lemma 2.

(6 \Rightarrow 3) Suppose that L is generated by the finite set X , and let H be the Lie subalgebra of L generated by X . If $\dim_K H = \infty$, then $\Omega_H(X, n+1) > \Omega_H(X, n)$ for all $n \in \mathbb{N}$, and hence $\Omega_L(X, n) \geq \Omega_H(X, n) \geq n$. By definition, this yields $L_p \dim L = \infty$, a contradiction. Thus $\dim_K H < \infty$, and the result follows since $L = H_p$.

(4 \Rightarrow 5, 6) are consequences of Lemma 3 below. \square

Lemma 3. *Let L be a restricted Lie algebra in characteristic p , and suppose that $L = R + Z$, where R is a finite-dimensional Lie subalgebra and Z is a central free abelian p -algebra of finite rank s . Then*

- (1) $u(L)$ is finitely generated of polynomial growth.
- (2) $\mathcal{H}_X(u(L), t)$ is a rational function for any finite set X that generates L .
- (3) $\text{GKdim } u(L) = \underline{\text{GKdim}} u(L) = s$.
- (4) $L_p \dim L = s$.

Proof. Since $L = R + Z$, it is clear that L is finitely generated and that $[L, L] = [R, R] \subseteq R$. Now let X be an arbitrary finite generating set for L , and choose an integer N so that $R \subseteq L^{(X, N)}$. Suppose $a \in L$. Then, by definition of the length function, there exist finitely many $b_j \in L$ with $l_L(X, b_j) \leq l_L(X, a)/p$ and with $a \in [L, L] + \langle b_1^p, \dots, b_k^p \rangle_K$. In particular, if $l_L(X, a) > N$, then since $[L, L] \subseteq R \subseteq L^{(X, N)}$, some b_j must satisfy $l_L(X, b_j) = l_L(X, a)/p$. Of course, this implies that $l_L(X, a)$ is divisible by p .

Consider the sequence of finite-dimensional vector spaces defined by $V_i = L^{(X, p^{i+1}N)} / L^{(X, p^i N)}$ for $i \geq 0$, and set $f(i) = \dim_K V_i$. Since $L^{(X, N)} \supseteq [L, L]$, it is clear that the p -power map defines a p -semilinear transformation $\theta_i: V_i \rightarrow V_{i+1}$ for all $i \geq 0$. Furthermore, by the above, we see that θ_i is essentially onto, that is $V_{i+1} = \langle \theta_i(V_i) \rangle_K = \langle V_i^p \rangle_K$. (Of course, the map would necessarily be onto if K were assumed to be perfect.) Thus $f(i+1) \leq f(i)$ and it follows that the sequence $f(0), f(1), \dots$ must stabilize. Without loss of generality, we may assume that $f(i) = f(0) = s'$ for all $i \geq 0$.

Next, we show that if $c \in L$ with $l_L(X, c) > N$, then $l_L(X, c^p) = l_L(X, c)p$. Say $Np^i < l_L(X, c) \leq Np^{i+1}$, and for each integer j with $0 \leq j \leq Np^{i+1} - Np^i$, define $V_{i,j} = L^{(X, Np^i + j)} / L^{(X, Np^i)} \subseteq V_i$ and $V_{i+1, pj} = L^{(X, Np^{i+1} + pj)} / L^{(X, Np^{i+1})} \subseteq V_{i+1}$. Then, by the observation of the first paragraph, we see that $\theta_i: V_i \rightarrow V_{i+1}$ maps $V_{i,j}$ essentially onto $V_{i+1, pj}$, so that $V_{i+1, pj} = \langle \theta_i(V_{i,j}) \rangle_K = \langle V_{i,j}^p \rangle_K$ and $\dim_K V_{i,j} \geq \dim_K V_{i+1, pj}$. Furthermore, θ_i induces an essentially onto map $V_i/V_{i,j} \rightarrow V_{i+1}/V_{i+1, pj}$, so $\dim_K V_i/V_{i,j} \geq \dim_K V_{i+1}/V_{i+1, pj}$. Thus since $\dim_K V_i = \dim_K V_{i+1}$, we see that $\dim_K V_{i,j} = \dim_K V_{i+1, pj}$, and it follows that an element of $V_{i,j} \setminus V_{i,j-1}$ must map to an element of $V_{i+1, pj} \setminus V_{i+1, pj-p}$. In particular, since $l_L(X, c) = Np^i + k$ for some $1 \leq k \leq Np^{i+1} - Np^i$, we conclude that $(Np^i + k)p \geq l_L(X, c^p) > (Np^i + k - 1)p \geq Np$. But $l_L(X, c^p)$ must be a multiple of p , so $l_L(X, c^p) = (Np^i + k)p = l_L(X, c)p$, as required. This argument also shows that θ_i maps a basis of V_i that is compatible with the X -length filtration of L to a basis of V_{i+1} that is also compatible with the X -length filtration. (It is worth noting that a one-to-one, essentially onto p -semilinear transformation need not preserve dimension over nonperfect fields.)

Since $R \subseteq L^{(X, N)}$ and $L = R + Z$, we can choose $\{z_1, \dots, z_{s'}\} \subseteq Z$ so that its image in V_0 is a basis compatible with the X -length filtration. By the result of the preceding paragraph, it follows that the image of $\{z_1^{p^k}, \dots, z_{s'}^{p^k}\}$ in V_k is a basis that is also compatible with the X -length filtration. In particular, $Z_1 = \langle z_1, \dots, z_{s'} \rangle_p$ is a central free abelian restricted subalgebra of L and $L = L^{(X, N)} \oplus Z_1$. Furthermore, Z_1 has

finite codimension in Z , so it is clear that $s' = s$. Note also, by the above, that if $d_i = l_L(X, z_i)$, then $d_i p^k = l_L(X, z_i^{p^k})$.

We now enter this information into equation (5). Since b_n , for $n > N$, is clearly equal to the number of ordered pairs (i, k) with $d_i p^k = n$, we see that $\mathcal{H}_X(u(L), t)$ is equal to

$$\begin{aligned} & \prod_{n=1}^N (1 + t^n + \dots + t^{(p-1)n})^{b_n} \prod_{i=1}^s \prod_{k=0}^{\infty} (1 + t^{d_i p^k} + \dots + t^{(p-1)d_i p^k}) \\ &= \prod_{n=1}^N (1 + t^n + \dots + t^{(p-1)n})^{b_n} \prod_{i=1}^s \prod_{k=0}^{\infty} \frac{1 - t^{d_i p^{k+1}}}{1 - t^{d_i p^k}} \\ &= \prod_{n=1}^N (1 + t^n + \dots + t^{(p-1)n})^{b_n} \prod_{i=1}^s \frac{1}{1 - t^{d_i}}. \end{aligned}$$

In particular, $\mathcal{H}_X(u(L), t)$ is rational, and we clearly have

$$\mathcal{H}(u(L), t) \approx \frac{C}{(1-t)^s}, \quad t \rightarrow 1^-$$

where

$$C = \frac{p^{b_1 + \dots + b_N}}{d_1 \dots d_s} = \frac{p^{\dim_K L/Z_1}}{d_1 \dots d_s}.$$

Furthermore, well-known facts on the growth of rational functions [11] now imply that

$$\begin{aligned} \lambda_{u(L)}(n) &\approx C_1 n^{s-1}, \quad n \rightarrow \infty \\ \Omega_{u(L)}(n) &\approx C_2 n^s, \quad n \rightarrow \infty \\ \text{GKdim } u(L) &= \underline{\text{GKdim}} u(L) = s. \end{aligned}$$

Finally, by construction, $N < d_i \leq pN$ for $i = 1, \dots, s$, so

$$\Omega_L(X, n) = \dim_K L^{(X, N)} + |\{(i, k) \mid 1 \leq i \leq s, d_i p^k \leq n\}|$$

when $n \geq N$. It follows that

$$s \lceil \log_p \frac{n}{pN} \rceil \leq \Omega_L(X, n) \leq \dim_K L^{(X, N)} + s(\lceil \log_p \frac{n}{N} \rceil + 1)$$

and therefore $L_p \dim L = s$. \square

These results also yield

Corollary 2. *Let L be a finitely generated restricted Lie algebra. Then*

$$\text{GKdim } u(L) = L_p \dim L = s.$$

Furthermore, if $s < \infty$, then s is the rank of a central free abelian p -algebra of finite codimension in L .

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