# Answers to the Algebra Qualifying Exam
## August 2000

1. a. Since $S, T \lhd G$, it follows that the commutator group $[S, T]$ is contained in both $S$ and $T$. Thus $[S, T] \subseteq S \cap T = 1$, and $S$ and $T$ commute elementwise. Now $S \lhd G$, so $S \cap H \lhd H$. Hence, since $T$ centralizes $S \cap H \subseteq S$, it follows that $\mathbb{N}_G(S \cap H) \supseteq HT = G$, and $S \cap H \lhd G$. The argument for $T \cap H$ is similar.

b. Now assume that $S \cap H = 1 = T \cap H$. Then $G/S = SH/S \cong H/(S \cap H) \cong H$. On the other hand, since $G$ is the internal direct product of $S$ and $T$, we have $G/S \cong T$. Thus $T \cong H$ and by symmetry, $S \cong H$. Therefore, $S \cong H \cong T$.

c. Finally, suppose in addition that $H \lhd G$. Then $[H, S] \subseteq S \cap H = 1$ and $[H, T] \subseteq T \cap H = 1$, so $H$ centralizes both $S$ and $T$. In particular, $\mathbb{C}_G(S) \supseteq HT = G$ and $\mathbb{C}_G(T) \supseteq HS = G$. Thus $\mathbb{Z}(G) \supseteq ST = G$, and $G$ is abelian.

2. a. Since $D \subseteq A_i$, it is clear that $\sqrt{D} \subseteq \sqrt{A_i}$ for all $i$, and hence $\sqrt{D} \subseteq \bigcap_{i=1}^n \sqrt{A_i}$. Conversely, suppose $r \in \bigcap_{i=1}^n \sqrt{A_i}$. Then $r \in \sqrt{A_i}$ so there exists an integer $m_i$ with $r^{m_i} \in A_i$. In particular, if $m$ is the largest of the integers $m_i$, then $r^m \in A_i$, since $A_i$ is an ideal. Hence $r^m \in \bigcap_{i=1}^n A_i = D$, and $r \in \sqrt{D}$. This yields the reverse inclusion.

b. Fix $i$. Then we already know that $\sqrt{A_i} \supseteq \sqrt{D}$. Conversely, let $r \in \sqrt{A_i}$ so that $r^m \in A_i$ for some integer $m$. Since $\bigcap_{j \neq i} A_j$ properly contains $D$, by assumption, we can choose an element $s$ in this intersection but not in $D$. Then

$$r^m s \in A_i \cap \bigcap_{j \neq i} A_j = \bigcap_{j=1}^n A_j = D.$$

But $D$ is primary and $s \notin D$, so it follows that some power $(r^m)^t$ of $r^m$ is contained in $D$. In other words, $r^{mt} \in D$, and hence $r \in \sqrt{D}$. This is the reverse inclusion, so $\sqrt{D} = \sqrt{A_i}$.

3. a. Let $n = |F_2 : K|$ and let $\{a_1, a_2, \ldots, a_n\}$ be a basis for $F_2$ over $K$. Then, for any subscripts $i, j$, we have $a_i a_j \in F_2 = \sum_k K a_k$. Thus, since $F_1 \supseteq K$, it follows that $F = \sum_k F_1 a_k$ is closed under both addition and multiplication. Hence $F \supseteq K$ is a subring of the algebraic closure of $K$, so $F$ is a field which is clearly equal to $\langle F_1, F_2 \rangle$. Since $\{a_1, a_2, \ldots, a_n\}$ spans $F$ over $F_1$, we have $|F : F_1| \le n = |F_2 : K|$.

b. Let $F = \langle F_1, F_2 \rangle$. By (a), $|F : K| = |F : F_1||F_1 : K| \le |F_2 : K||F_1 : K|$. Next, since $K \subseteq F_i \subseteq F$, we see that $|F_i : K|$ divides $|F : K|$ for $i = 1, 2$. But these two factors are relatively prime, so $|F_2 : K||F_1 : K|$ divides $|F : K|$, and hence $|F_2 : K||F_1 : K| \le |F : K|$. Since this is the reverse inequality, we must have equality throughout, and therefore $F_1$ and $F_2$ are linearly disjoint over $K$.

c. Take $K = Q$, the rationals, and let $F_1 = Q[\sqrt{2}]$ and $F_2 = Q[i]$, where $i = \sqrt{-1}$. Then both $F_1$ and $F_2$ have degree 2 over $Q$. Furthermore, since $F_1$ is real, we have $i \notin F_1$, and hence $F = F_1[i]$ has degree 2 over $F_1$. Since $F = Q[\sqrt{2}, i] = \langle F_1, F_2 \rangle$, we see that $|F : Q| = |F : F_1||F_1 : Q| = 2|F_1 : Q| = |F_2 : Q||F_1 : Q|$, so $F_1$ and $F_2$ are linearly disjoint.

d. Let $F = \langle F_1, F_2 \rangle$. Since $F_i$ is Galois over $K$, for $i = 1, 2$, we know that restriction determines a group homomorphism $\mathrm{Gal}(F/K) \to \mathrm{Gal}(F_i/K)$. By combining these two, we obtain a homomorphism $\theta\colon \mathrm{Gal}(F) \to \mathrm{Gal}(F_1) \times \mathrm{Gal}(F_2)$. Furthermore, $\theta$ is one-to-one. Indeed, if $\sigma \in \mathrm{Gal}(F/K)$ is in the kernel of $\theta$, then it acts like the identity on $F_1$ and $F_2$, and hence it fixes $\langle F_1, F_2\rangle = F$. Thus $\sigma = 1$ and $\theta$ is one-to-one. Finally, since $F_1$ and $F_2$ are Galois over $K$, it follows that $F$ is also Galois over $K$. Thus

$$|\mathrm{Gal}(F/K)| = |F : K| = |F_1 : K||F_2 : K| = |\mathrm{Gal}(F_1/K) \times \mathrm{Gal}(F_2/K)|$$

and, since $\theta$ is one-to-one, it must be onto.

4. a. Let $K = \{v \in V \mid Av = 0\}$. Then $K$ is a subspace of $V$ with $AK = 0 \subseteq K$. Furthermore, if $v \in K$, then $A(Bv) = \lambda B(Av) = \lambda B(0) = 0$, so $Bv \in K$. Thus $AK \subseteq K$, $BK \subseteq K$ so, by assumption, $K = 0$ or $K = V$. But $K = V$ implies that $A = 0$, a contradiction. Therefore, $K = 0$ and $A$ is one-to-one. Next, let $I = \{Av \mid v \in V\}$ so that $I$ is also a subspace of $V$. Clearly $A(Av) \in I$ and $B(Av) = \lambda^{-1}A(Bv) = A(\lambda^{-1}Bv) \in I$. Thus $AI \subseteq I$, $BI \subseteq I$ so, by assumption, $I = 0$ or $I = V$. But $I = 0$ implies that $A = 0$, a contradiction. So $I = V$ and $A$ is onto. Using $BA = \lambda^{-1}AB$, we conclude in a similar manner that $B$ is one-to-one and onto.

b. Say $\dim V = n$ and fix a basis for $V$. If $[A]$ denotes the matrix of $A$ with respect to this basis, then $[A][B] = \lambda[B][A]$ and hence, taking determinants yields $\det[A]\det[B] = \lambda^n \det[B]\det[A]$. Since $A$ and $B$ are one-to-one and onto, by (a), their determinants are not zero and we conclude that $\lambda^n = 1$.

c. In view of (b), we should try $n = 2$ when $\lambda = -1$. Also the matrices involved must be nonsingular. A little thought yields the matrices

$$[A] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{and} \qquad [B] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Finally, take $A$ and $B$ to be the corresponding transformations.

5. a. We have

$$A_r(a+b) = (a+b)r - r(a+b) = (ar - ra) + (br - rb) = A_r(a) + A_r(b)$$

$$A_r(ab) = (ab)r - r(ab) = a(br - rb) + (ar - ra)b = aA_r(b) + A_r(a)b$$

so $A_r$ is a derivation.

b. Let $z \in Z$ and $r \in R$. Then $rz = zr$ so

$$rD(z) + D(r)z = D(rz) = D(zr) = zD(r) + D(z)r.$$

Thus since $z$ commutes with $D(r)$, we get $rD(z) = D(z)r$. Hence $D(z)$ commutes with $r$, for all $r \in R$, so $D(z) \in Z$.

c. Note that $(1-2e)^2 = 1 - 4e + 4e^2 = 1 - 4e + 4e = 1$. In the following, we use the fact that $e$ commutes with $D(e)$. This holds since $e \in Z$. Now $e = e^2$, so

$$D(e) = D(e^2) = eD(e) + D(e)e = 2eD(e).$$

Thus $(1-2e)D(e) = 0$ and, by multiplying on the left by $(1-2e)$, we get $D(e) = 0$.