Answers to Algebra Qualifying Exam
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1. We are to assume that $\{p, q\} \neq \{2, 3\}$. Suppose $p < q$, let P be a Sylow p-subgroup of G and let N be its normalizer. By assumption $|G : N| = p + 1$. If $q|(p + 1)$, then $q > p$ yields $q = p + 1$ and $\{p, q\} = \{2, 3\}$, contradiction. Hence q does not divide $p + 1$ and N contains a full Sylow q-subgroup Q of G. Thus $H = PQ$ is a subgroup of G with $P \triangleleft H$. The number of Sylow q-subgroups of H is $\equiv 1\ (q)$ and less than the number in G. Thus either $Q \triangleleft H$ or H has $q + 1$ Sylow q-subgroups. In the latter case, since $|H : Q| = |P|$ is a power of p, we obtain $q + 1 = p^a$, so $q = p^a - 1 = (p - 1)(p^{a-1} + \cdots + 1)$. Since q is prime and $q > p$, this is a contradiction. Thus both P and Q are normal in H, so $H = P \times Q$.

2. (i) Since R is Noetherian, choose $0 \neq a \in R$ so that $\mathrm{ann}(a)$ is maximal in the set $\{\mathrm{ann}(b) \mid 0 \neq b \in R\}$. We claim that $\mathrm{ann}(a)$ is prime. Thus suppose $I,\ J$ are ideals of R properly containing $\mathrm{ann}(a)$. By definition, I and J cannot annihilate nonzero elements of R. Thus $aI \neq 0$ and then $(aI)J \neq 0$. Thus $IJ \not\subseteq \mathrm{ann}(a)$ so $\mathrm{ann}(a)$ is prime and $a \in S$.

(ii) Suppose $ar \neq 0$ and let $IJ \subseteq \mathrm{ann}(ar)$. Then $arIJ = 0$ so $rIJ \subseteq \mathrm{ann}(a)$. But $\mathrm{ann}(a)$ is prime and $r \notin \mathrm{ann}(a)$, so either $I \subseteq \mathrm{ann}(a)$ or $J \subseteq \mathrm{ann}(a)$. In the former case, $rI \subseteq \mathrm{ann}(a)$ so $arI = 0$ and $I \subseteq \mathrm{ann}(ar)$. In the latter case, $J \subseteq \mathrm{ann}(ar)$ so $\mathrm{ann}(ar)$ is prime and $ar \in S$.

(iii) Assume $ab \neq 0$. Clearly $\mathrm{ann}(ab) \supseteq \mathrm{ann}(a)$. Conversely, $a \cdot b\,\mathrm{ann}(ab) = 0$ so $b\,\mathrm{ann}(ab) \subseteq \mathrm{ann}(a)$. Since $\mathrm{ann}(a)$ is prime and $b \notin \mathrm{ann}(a)$, we have $\mathrm{ann}(ab) \subseteq \mathrm{ann}(a)$. Thus $\mathrm{ann}(ab) = \mathrm{ann}(a)$ and similarly $\mathrm{ann}(ab) = \mathrm{ann}(b)$. This is a contradiction since $\mathrm{ann}(a) \neq \mathrm{ann}(b)$.

3. It is clear that $E \supseteq F$. Suppose $\alpha, \beta \in E$ so that $\mathrm{Gal}(F[\alpha]/F)$ and $\mathrm{Gal}(F[\beta]/F)$ are abelian. Since $F[\alpha]$ and $F[\beta]$ are splitting fields of the separable polynomials $f(x), g(x) \in F[x]$, respectively, it follows that $F[\alpha, \beta]$ is the splitting field of the separable polynomial $f(x)g(x)$. Thus $F[\alpha, \beta]$ is Galois over F. If $\sigma, \tau \in \mathrm{Gal}(F[\alpha, \beta]/F)$, then σ and τ commute when restricted to $F[\alpha]$ and to $F[\beta]$. Thus the commutator $\sigma^{-1}\tau^{-1}\sigma\tau$ fixes both α and β, so it is the identity. This shows that $G = \mathrm{Gal}(F[\alpha, \beta]/F)$ is abelian. Finally, let $\gamma \in F[\alpha, \beta]$. Then $F[\gamma] = F[\alpha, \beta]^H$ for some subgroup H of G. Since G is abelian, $H \triangleleft G$. Thus $F[\gamma]$ is Galois over F with $\mathrm{Gal}(F[\gamma]/F) = G/H$ abelian. This shows that $\gamma \in E$. In other words, if $\alpha, \beta \in E$, then so are $\gamma = \alpha + \beta,\ \alpha\beta$ and α^{-1} (if $\alpha \neq 0$). Thus E is a field.

4. (i) Let $v \in V$. Since T is nilpotent, choose k minimal with $vT^k = 0$. If $k \geq 2$, set $w = vT^{k-2}$ so that $wT \in N$. Then $(wT, N) = (w, NT) = (w, 0) = 0$ and since $(\ ,\ )$ is nondegenerate on N, we conclude that $0 = wT = vT^{k-1}$, a contradiction. Thus $k \leq 1$ and $vT = 0$.

(ii) Let V be the 2-dimensional vector space with basis $\{v, w\}$ and let $T : V \to V$ be given by $vT = w,\ wT = 0$. Thus T is nilpotent but not zero. Define the symmetric bilinear form $(\ ,\ )$ so that $(v, v) = (w, w) = 0$ and $(v, w) = 1$. It is easy to verify that $(\ ,\ )$ is nonsingular, since no nonzero element of V is orthogonal to both v and w. Furthermore, by linearity, we can verify that $(xT, y) = (x, yT)$ by considering the four cases with $x, y \in \{v, w\}$. Indeed, when $x = y$ there is nothing to check, so by symmetry, we need only consider $x = v,\ y = w$.

5. Consider the chain $0 \subseteq X \subseteq X+Y \subseteq M$ and extend this to a composition series of M. Notice that $(X+Y)/X \cong Y/(X \cap Y)$ so that the composition factors between $X+Y$ and X are factors of Y and hence of $X \cong Y$. In particular, if $X+Y \neq X$, then a composition factor between $X+Y$ and X will also show up in the series between 0 and X. This means that M has two composition factors which are isomorphic and this contradicts the hypothesis and the Jordan-Holder theorem. Thus $X+Y=X$. Similarly $X+Y=Y$ and therefore $X=Y$.