Answers to Algebra Qualifying Exam
August 1994

1. (i) We are given $N \triangleleft G$ with $|G:N| = |P|$. Thus $|G| = |N|\,|P|$, so $|N| = |G|/|P|$ is a number prime to p. By Lagrange's theorem all elements of N have orders prime to p and also $N \cap P = 1$. The latter implies that $|NP| = |G|$ so $G = NP$ and $G/N = NP/N \cong P/(P\cap N) \cong P$. In particular, if g is a p'-element, then its image $\bar{g} \in G/N$ is a p'-element. But G/N is a p-group, so $\bar{g}$ is also a p-element. Hence $\bar{g} = 1$ and $g \in N$.

(ii) Let q be a prime different from p and let Q be a Sylow q-subgroup of $\mathbb{N}_G(P)$. By (i), $Q \subseteq N$ and, since $N \triangleleft G$, the commutator $[Q,P]$ is contained in N. On the other hand, Q normalizes P, so $[Q,P] \subseteq P$. Thus $[Q,P] \subseteq N \cap P = 1$ and hence Q and P centralize each other. If $Q \neq 1$, then since $P \neq 1$, we can choose $x \in Q$ of order q and $y \in P$ of order p. Since $xy = yx$, it follows that xy has order qp and hence $xy \notin N$. But all elements outside of N have p-power order, and this is a contradiction. Thus $Q = 1$ and it follows that $\mathbb{N}_G(P)$ is a p-group. But $\mathbb{N}_G(P) \supseteq P$ and P is a maximal p-subgroup of G, so $\mathbb{N}_G(P) = P$.

2. (i) We have $W = \{\, v \in V \mid va = 0 \text{ for some } a \in R \setminus P \,\}$ and we note that $0 \in W$ since $0 \cdot 1 = 0$. If $v_1, v_2 \in W$ with $v_i a_i = 0$, then $(v_1 + v_2)a_1a_2 = 0$ and $a_1a_2 \in R \setminus P$ since P is prime. Thus $v_1 + v_2 \in W$. Furthermore, if $r \in R$, then $(v_1 r)a_1 = 0$ so $v_1 r \in W$ and W is a submodule of V.

(ii) If R is Noetherian and V is finitely generated, then V is a Noetherian R-module and hence all submodules are finitely generated. In particular, $W = w_1R + \cdots + w_nR$ for suitable $w_1, \ldots, w_n \in W$. If $w_ib_i = 0$ with $b_i \in R \setminus P$, then $b = b_1 \cdots b_n \in R \setminus P$, since P is prime, and it follows as in (i) that $w_ib = 0$. Thus $Wb = (\sum_i w_iR)b \subseteq \sum_i (w_ib)R = 0$.

(iii) Fix $0 \neq v \in V$. Since V is a simple R-module, the map $R \to V$ given by $r \mapsto vr$ is an R-module epimorphism. Thus $M = \{\, r \in R \mid vr = 0 \,\}$ is a maximal ideal of the commutative ring R. But $v \notin W = 0$, so $M \cap (R \setminus P) = \emptyset$. Thus $M \subseteq P$ and, since M is maximal, we have $P = M$ maximal.

3. (i) It is clear that $E = \mathbb{Q}[\sqrt[3]{2}, \omega] = F[\omega]$ where $\omega = \frac{-1+\sqrt{-3}}{2}$ is a primitive complex cube root of 1. Thus $|E:F| = 2$ and, since $|F:\mathbb{Q}| = 3$ by Eisenstein's criterion, we have $|E:\mathbb{Q}| = 6$. If $G = \mathrm{Gal}(E/\mathbb{Q})$, then $|G| = 6$ and, since G faithfully permutes the 3 roots of $x^3 - 2$, we have $G \cong \mathrm{Sym}_3$. Choose $\sigma \in G$ of order 3 with $\sigma(\sqrt[3]{2}) = \omega\sqrt[3]{2}$ and let L be its fixed field. Then $|E:L| = 3$, so $|L:\mathbb{Q}| = 2$. But $|F:\mathbb{Q}|$=3 and therefore we have $F \cap L = \mathbb{Q}$ since $|F \cap L : \mathbb{Q}|$ divides both 2 and 3. Note that $F \cap L = \mathbb{Q}$ says that any element of F which is fixed by σ is rational.

(ii) Since $a^3 \in \mathbb{Q}$, $\sigma(a)$ is a root of $x^3 - a^3$ and hence $\sigma(a) = a\omega^{-i}$ for some $i = 0, 1, 2$. Thus σ fixes $a(\sqrt[3]{2})^i \in F$ and part (i) implies that $a(\sqrt[3]{2})^i \in \mathbb{Q}$.

(iii) If $a = \sqrt[3]{3} \in E$, then a is in F since it is real. Also $a^3 \in \mathbb{Q}$, so (ii) implies that one of a, $a\sqrt[3]{2}$, $a\sqrt[3]{4}$ is in $\mathbb{Q}$. In other words, $\sqrt[3]{3}$, $\sqrt[3]{6}$ or $\sqrt[3]{12}$ is in $\mathbb{Q}$ and this contradicts Eisenstein's criterion.

4. Let $\tilde{K}$ be the algebraic closure of K and embed $\mathrm{M}_n(K)$ into $\mathrm{M}_n(\tilde{K})$. Since the characteristic polynomial of A has distinct roots in $\tilde{K}$, A is similar in the larger matrix ring to a diagonal matrix with distinct diagonal entries. In other words, $P^{-1}AP = D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with all λ_i distinct. If X and Y commute with A, then $P^{-1}XP$ and $P^{-1}YP$ commute with D. Furthermore, by considering the equation $ZD = DZ$ in $\mathrm{M}_n(\tilde{K})$, it follows easily that the centralizer of D must be diagonal. Indeed, if $Z = [z_{i,j}]$, then the i,jth entry of ZD is $z_{i,j}\lambda_j$ while the i,jth entry of $DZ = ZD$ is $z_{i,j}\lambda_i$. Thus, if $i \neq j$, then $\lambda_i \neq \lambda_j$ implies that $z_{i,j} = 0$ and Z is diagonal. In particular, $P^{-1}XP$ and $P^{-1}YP$ are diagonal, so they commute, and conjugation by P^{-1} implies that X and Y commute.

5. (i) If s is a nilpotent matrix and if λ is an eigenvalue of s, then λ is a nilpotent field element, so $\lambda = 0$. Since the trace of s is the sum of its eigenvalues, we have $\mathrm{tr}\, s = 0$.

(ii) Since the trace is additive and $\theta(I)$ is the additive span of nilpotent matrices, we know that $\mathrm{tr}\, \theta(I) = 0$ from (i). Also, since θ is surjective, it follows that $\theta(I)$ is an ideal of the ring S. But $\mathrm{tr}\, \theta(I) = 0$, so $\theta(I)$ cannot contain the matrix unit $e_{1,1}$ and hence $\theta(I) \neq S$. Thus $\theta(I) = 0$ since S is a simple ring.