Answers to Algebra Qualifying Exam
August 1995

1. (i) If M does not contain the center Z, then $MZ > M$ and, since M is maximal, we have $MZ = G$. But then $M \triangleleft MZ = G$ and this is a contradiction. Thus $M \supseteq Z$ and $N \supseteq Z$ so $M \cap N \supseteq Z$. Conversely, since M and N are abelian, $\mathbb{C}_G(M \cap N) \supseteq \langle M, N \rangle$ and certainly $\langle M, N \rangle = G$ since M and N are maximal and distinct. Thus $M \cap N \subseteq Z$.

(ii) For any $x \in G$, it is clear that M^x also has property $(*)$. Thus if M^x and M^y are distinct, then $M^x \cap M^y = Z$. Now $|M^x| = |M|$, so it follows that $|S(M)| = (|M| - |Z|)|G : \mathbb{N}_G(M)|$ since $|G : \mathbb{N}_G(M)|$ counts the number of conjugates of M. But $M \subseteq \mathbb{N}_G(M) < G$, since M is not normal, so since M is maximal, we have $\mathbb{N}_G(M) = M$ and $|S(G)| = (m - z)g/m$. Note that $(m - z)g/m = g - z(g/m) < g - z$ since $g/m > 1$. Also $M \supseteq Z$ and M not normal, so $M > Z$ and $m \geq 2z$. Thus

$$(m - z)g/m = g(1 - z/m) \geq g(1 - 1/2) > (1/2)(g - z).$$

(iii) If M and N are not conjugate, then it follows from (i) that $S(M)$ and $S(N)$ are disjoint subsets of $G \setminus Z$. Thus

$$g - z = |G \setminus Z| \geq |S(M)| + |S(N)| > (1/2)(g - z) + (1/2)(g - z) = g - z,$$

a contradiction.

2. (i) Since V satisfies the minimal condition, we can choose V_0 to be a submodule of V minimal with the property that $V \cong V_0$. Thus V_0 has no proper submodule isomorphic to V, and therefore the same is true of V. By assumption, $V \supseteq W_1$ with $W_1 \cong W$ and $W_1 \supseteq V_1$ with $V_1 \cong V$. But then $V \supseteq W_1 \supseteq V_1$, so $V = V_1 = W_1$ and $V \cong W$.

(ii) By the fundamental theorem of abelian groups, $V = T(V) + \mathbb{Z}^r$ where $T(V)$ is the set of elements of V of finite order, and where $\mathbb{Z}^r$ is the direct sum of r copies of $\mathbb{Z}$. Also $T(V)$ is finite and r is called the rank of V. If $V \supseteq W$, then certainly $T(V) \supseteq T(W)$ and hence $|T(V)| \geq |T(W)|$. In particular, if $V \sim W$, then $|T(V)| = |T(W)|$ and if $V \supseteq W$, then $T(V) = T(W) = T$, say. Then $\mathbb{Z}^r = V/T \supseteq W/T = \mathbb{Z}^s$, where r is the rank of V and s is the rank of W. For various reasons (for example, tensor with the rationals $\mathbb{Q}$ and note that the rank becomes equal to the dimension) it follows that $r \geq s$. Similarly, $s \geq r$, so $s = r$ and $V \cong W$.

(iii) If I is a nonzero ideal of R, then certainly I is isomorphic to a submodule of R. Also if $0 \neq a \in I$, then since R is a domain, $I \supseteq aR \cong R$. Thus $I \sim R$. Note that if $I \cong R$, then I is principal. In particular, if R is not a PID, then we can choose I not principal, and hence $I \sim R$ but $I \not\cong R$.

3. (i) For convenience, write $F_2 = \mathbb{Q}[\sqrt{2}]$ and $F_3 = \mathbb{Q}[\sqrt[3]{2}]$ for the corresponding real fields. Since the polynomials $x^2 - 2$ and $x^3 - 2$ are irreducible over $\mathbb{Q}$ by Eisenstein's criterion, it follows that $|F_2 : \mathbb{Q}| = 2$ and $|F_3 : \mathbb{Q}| = 3$. In particular, since E contains F_2 and F_3 and since 2 and 3 are relatively prime, it follows that $|E : \mathbb{Q}|$ is divisible by $2 \cdot 3 = 6$. On the other hand, $E = F_3[\sqrt{2}]$, so $|E : F_3| \leq 2$ and hence $|E : \mathbb{Q}| = |E : F_3||F_3 : \mathbb{Q}| \leq 6$. Thus $|E : \mathbb{Q}| = 6$.

(ii) If $\mathbb{Q} \subseteq K \subseteq E$, then $|K : \mathbb{Q}|$ divides $|E : \mathbb{Q}| = 6$ and thus $|K : \mathbb{Q}| = 1, 2, 3$ or 6. If $|K : \mathbb{Q}| = 1$, then $K = \mathbb{Q}$ and if $|K : \mathbb{Q}| = 6$, then $K = E$. Suppose $|K : \mathbb{Q}| = 2$ and let $L = K[\sqrt{2}]$. Then $|L : K| \leq 2$, so $|L : \mathbb{Q}|$ is even and at most 4. But $|L : \mathbb{Q}|$ divides 6, so this implies that $|L : \mathbb{Q}| = 2$. Hence $L = K$, so $\sqrt{2} \in K$ and $K \supseteq F_2$. In particular, since they have the same degree over $\mathbb{Q}$, it follows that $K = F_2$.

Finally, let $|K : \mathbb{Q}| = 3$. Then $|E : K| = 2$ and, since the fields have characteristic 0, this clearly implies that E/K is Galois with group $\{1, \sigma\}$ of order 2. Note that $(\sqrt[3]{2})^\sigma$ is also a root of $x^3 - 2$. But $x^3 - 2$ has only one real root and $(\sqrt[3]{2})^\sigma \in E$, a real field. Thus $(\sqrt[3]{2})^\sigma = \sqrt[3]{2}$ and $\sqrt[3]{2} \in K$, the fixed field. Thus $K \supseteq F_3$ and since they have the same degrees, we conclude that $K = F_3$.

(iii) Let $L = \mathbb{Q}[\sqrt{2} + \sqrt[3]{2}]$ so that $\mathbb{Q} \subseteq L \subseteq E$ and note that there are only four possibilities for L. If $L \subseteq F_2$, then $\sqrt{2}$ and $\sqrt{2} + \sqrt[3]{2}$ are in F_2, so $F_2 \supseteq \mathbb{Q}[\sqrt{2}, \sqrt[3]{2}] = E$, a contradiction. Similarly, L cannot be contained in F_3. Thus, by (ii), $L = E$.

4. (i) Note that $T(A(N)) = B(T(N)) = B(0) = 0$ and thus $A(N) \subseteq \ker T = N$.

(ii) Part (i) implies that under the composite map $V \xrightarrow{A} V \to V/N$, the subspace N maps to 0. Thus we obtain a map $\bar{A} : V/N \to V/N$. Note that $V \neq N$ since $T \neq 0$. Hence, since F is algebraically closed, $\bar{A}$ has a (nonzero) eigenvector $\bar{v} \in V/N$ with $\bar{A}(\bar{v}) = \lambda\bar{v}$ for some $\lambda \in F$. If $\bar{v} = v + N$, then $v \notin N$ and $A(v) - \lambda v \in N$.

(iii) Note that $A(v) - \lambda v = n \in N$, so

$$B(T(v)) = T(A(v)) = T(\lambda v + n) = \lambda T(v) + T(n) = \lambda T(v)$$

since $T(n) = 0$. Also $v \notin N$, so $T(v) \neq 0$. Thus λ is an eigenvalue for B with eigenvector $T(v) \neq 0$. Also $(A - \lambda I)(v) \in N$ implies that $A - \lambda I$ is singular on V/N. Thus it must be singular on V and, if $0 \neq v'$ is in its kernel, then v' is an eigenvector for A with the same eigenvalue λ.

5. (i) For convenience write $(a, b) = \begin{bmatrix} a & \bar{b} \\ b & \bar{a} \end{bmatrix}$. Then $(a, b) + (c, d) = (a + c, b + d) \in S$ and $(a, b)(c, d) = (ac + \bar{b}d, bc + \bar{a}d) \in S$. Thus S is a subring of $\mathrm{M}_2(\mathbb{C})$ with identity $(1, 0)$.

(ii) Now suppose that (a, b) is in the center Z of S. Then (a, b) must commute with the diagonal matrix $(i, 0)$ where of course $i = \sqrt{-1}$, and this implies easily that $b = 0$. Moreover $(a, 0) = (a, b)$ commutes with $(0, i)$ and we conclude that $a \in \mathbb{R}$. Thus $Z \subseteq (\mathbb{R}, 0)$. On the other hand, it is easy to see that $(\mathbb{R}, 0)$ is central in S since it consists of scalar matrices in $\mathrm{M}_2(\mathbb{C})$. Thus $Z = (\mathbb{R}, 0) \cong \mathbb{R}$.

(iii) Here $I = \{(x, x) \mid x \in \mathbb{C}\}$ and by (i), I is clearly closed under addition. Furthermore, $(x, x)(c, d) = (xc + \bar{x}d, xc + \bar{x}d) \in I$, so I is a right ideal of S. Note that, if $(x, x) \neq 0$, then $(x, x)(c, 0) = (xc, xc)$ fills all of I and thus I is a minimal right ideal of S. Finally, if $I(c, d) = 0$, then $0 = (1, 1)(c, d)$ implies that $c + d = 0$ while $0 = (i, i)(c, d)$ implies that $c - d = 0$. Thus $c = d = 0$ and S acts faithfully on I.

(iv) It is clear that $\dim_Z I = 2$ with basis $\{(1, 1), (i, i)\}$. Thus since S acts faithfully as Z-linear transformations on I, we have a faithful embedding $\theta : S \to \mathrm{M}_2(Z)$, the ring of 2×2 matrices over Z. But it is easy to see that $\dim_Z S = 4$ with basis $\{(1, 0), (i, 0), (0, 1), (0, i)\}$ so θ must also be onto. Thus $S \cong \mathrm{M}_2(Z) \cong \mathrm{M}_2(\mathbb{R})$.