Answers to the Algebra Qualifying Exam
August 1997

1. a. The second Sylow theorem implies that $|G : N| = 50$. Suppose that H is a subgroup of G with $N \subseteq H \subseteq G$. Then P is a Sylow 7-subgroup of H and $N = \mathbb{N}_H(P)$, so $|H : N| \equiv 1\ (7)$. Also $|H : N|$ divides $|G : N| = 50$, and it is easy to check that the only possibilities here are $|H : N| = 1$ or 50. Thus $H = N$ or $H = G$, and consequently N is maximal in G.

b. Let Q be the given normal Sylow 5-subgroup of N. If R is a Sylow 5-subgroup of G containing Q, then $R \cap N = Q$ and R is properly larger than Q since $|G : N|$ is divisible by 5. We know that normalizers grow in p-groups and therefore $\mathbb{N}_R(Q)$ is properly larger than Q. Hence $\mathbb{N}_R(Q) \nsubseteq N$ since otherwise it would be contained in $R \cap N = Q$. Thus $\langle \mathbb{N}_R(Q), N\rangle$ is properly larger than N and, since N is maximal, we have $\langle \mathbb{N}_R(Q), N\rangle = G$. But both of these two generators of G normalize Q, so G normalizes Q and $Q \triangleleft G$.

2. a. By assumption, (ab) is primary, and certainly $ab \in (ab)$. Thus either $a \in (ab)$ or $b^n \in (ab)$ for some integer n. In the first case, $a = abr$ and, since $a \neq 0$ and R is a domain, we get $1 = br$ and b is a unit, contradiction. Thus $b^n \in (ab) \subseteq (a)$, as required.

b. Let $P \neq 0, R$ be a prime ideal of R and fix $0 \neq a \in P$. Now let b be any nonunit of R. Since the principal ideal (ab) is primary, by assumption, we deduce from the preceding part that $b^n \in (a) \subseteq P$. Furthermore, since P is prime, $b^n \in P$ yields $b \in P$. Thus P contains all the nonunits of R. On the other hand, any element of P must be a nonunit, since otherwise $P = R$. Thus P is precisely equal to the set of all nonunits of R and therefore P is the unique nonzero prime ideal of R.

3. a. Say $\alpha^n = a \in \mathbb{Q}$. Then α satisfies $x^n - a \in \mathbb{Q}[x]$, and consequently $g(x)$ divides $x^n - a$. Since the roots of $x^n - a$ are all of the form $\epsilon\alpha$ with ϵ a root of unity, it follows that the same is true of the roots of $g(x)$. Since the constant coefficient of $g(x)$ is $\pm$ the product of all roots, we conclude that $g(0) = \delta\alpha^m$ where δ is also a root of unity. But $g(0)$ and α are real, so δ is real and hence $\delta = \pm 1$. In other words, $g(0) = \pm\alpha^m$.

b. It follows from the above that $\alpha^m = b \in \mathbb{Q}$ and hence α is a root of $x^m - b$. Again we see that $g(x)$ divides $x^m - b$. Thus, since $g(x)$ is monic and has degree m, we conclude that $g(x) = x^m - b$.

c. Let $n = mq + r$ where q and r are integers with $0 \leq r < m$. Since α^n and α^m are both in $\mathbb{Q}$, it follows that $c = \alpha^r = \alpha^n/(\alpha^m)^q \in \mathbb{Q}$. If $r \geq 1$, then α is a root of $x^r - c$ and again we see that $g(x)$ divides $x^r - c$. But this is a contradiction since $\deg g(x) = m > r = \deg(x^r - c)$. Thus $r = 0$ and m divides n.

4. Suppose first that V has a basis $\{v_1, v_2, \ldots, v_n\}$ of eigenvectors of T with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v = \sum_i k_i v_i$ be an arbitrary vector of V. Then, for any $\lambda \in K$, we have $(\lambda I - T)v = \sum_i k_i(\lambda - \lambda_i)v_i$ and $(\lambda I - T)^2 v = \sum_i k_i(\lambda - \lambda_i)^2 v_i$. Since $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, it follows that v is in $\ker(\lambda I - T)$ or in $\ker(\lambda I - T)^2$ if and only if $k_i = 0$ for all subscripts i with $\lambda \neq \lambda_i$. Hence the two kernels are equal.

Conversely, assume both kernels are equal for all $\lambda \in K$. Let $f(x) = \prod_{i=1}^m (x - \lambda_i)$ be the minimal polynomial of T, and note that this polynomial splits since K is algebraically closed. Suppose two roots of $f(x)$ are equal, say $\lambda_1 = \lambda_2$, and set $g(x) = \prod_{i=3}^m (x - \lambda_i)$. Then $(\lambda_1 I - T)^2 g(T)V = f(T)V = 0$, so $g(T)V \subseteq \ker(\lambda_1 I - T)^2 = \ker(\lambda_1 I - T)$, by assumption. Hence $(T - \lambda_1 I)g(T)V = (\lambda_1 I - T)g(T)V = 0$ and T satisfies the polynomial $(x - \lambda_1)g(x)$ of degree $m - 1 < \deg f(x)$. But $f(x)$ is the minimal polynomial, so this cannot occur. It follows that $f(x)$ has distinct roots and splits. As is well known, this implies that V has a basis of eigenvectors of T.

(Indeed, if the roots of $f(x)$ are distinct, then it follows that the polynomials $f_i(x) = f(x)/(x - \lambda_i)$ are relatively prime. Thus, there exist $h_i(x)$ with $1 = \sum_i f_i(x)h_i(x)$, and hence $V = \sum_i f_i(T)h_i(T)V$. Note that $(T - \lambda_i I)\cdot f_i(T)h_i(T)V = f(T)h_i(T)V = 0$, so $f_i(T)h_i(T)V$ is contained in the eigenspace $\mathrm{E}(T, \lambda_i) = \{w \in V \mid Tw = \lambda_i w\}$. Hence, we have $V = \sum_i \mathrm{E}(T, \lambda_i)$, and it is easy to see that this sum is direct.)

5. a. Since θ is an R-module homomorphism, it is easy to check that W_θ is a submodule of V. Note that, for any $x \in X$, we have $x = (x - \theta(x)) + \theta(x) \in W_\theta + Y$. Hence, both X and Y are contained in $W_\theta + Y$, so $V = W_\theta + Y$. Furthermore, if $y' \in W_\theta \cap Y$, then $y' = x' - \theta(x')$ for some $x' \in X$, so $x' = y' + \theta(x') \in X \cap Y = 0$. Thus $x' = 0$ and $y' = y' + \theta(x') = 0$, so $W_\theta \cap Y = 0$ and hence $V = W_\theta \dotplus Y$.

b. Conversely, assume that $V = U \dotplus Y$. Since $X \subseteq V = U \dotplus Y$, we can let $\theta\colon X \to Y$ denote the projection map into the second coordinate. Clearly, θ is an R-homomorphism. Since $V = U \dotplus Y$, it follows that for any $x \in X$, we have $x = u + y = u + \theta(x)$ and thus $x - \theta(x) = u \in U$. In other words, $W_\theta \subseteq U$. For the reverse inclusion, let $u' \in U \subseteq X \dotplus Y$ and write $u' = x' - y'$ with $x' \in X$ and $y' \in Y$. Then $x' = u' + y' \in U \dotplus Y$ so, by the uniqueness of expression, we have $y' = \theta(x')$ and $u' = x' - \theta(x') \in W_\theta$. Thus $U \subseteq W_\theta$, as required, and the two submodules are equal.