Answers to the Algebra Qualifying Exam
August 1999

1. (a) One can almost just say that since H normalizes G and K, it must normalize $\mathbb{C}_G(K)$. More properly, let $g \in \mathbb{C}_G(K)$, $k \in K$, $h \in H$. Then $gk = kg$, so conjugating by h yields $g^h k^h = k^h g^h$. But $K \triangleleft H$, so $K^h = K$ and hence k^h is a typical element of K. It follows that g^h centralizes K, so $g^h \in \mathbb{C}_G(K)$ and hence $\mathbb{C}_G(K)^h \subseteq \mathbb{C}_G(K)$. Thus H normalizes $\mathbb{C}_G(K)$.

(b) It follows from (a) that $\bar{G} = H\mathbb{C}_G(K)$ is a group and that $\mathbb{C}_G(K) \triangleleft \bar{G}$. Furthermore, since $H \triangleleft G$ we have $H \triangleleft \bar{G}$. Note that $H \cap \mathbb{C}_G(K) = \mathbb{C}_H(K) = \langle 1 \rangle$ by assumption. Thus H and $\mathbb{C}_G(K)$ are disjoint normal subgroups of $\bar{G}$, so these groups commute elementwise and hence H centralizes $\mathbb{C}_G(K)$.

2. (a) Since Q is not prime there exist ideals A', B' with $A' \not\subseteq Q$, $B' \not\subseteq Q$ and $A'B' \subseteq Q$. Set $A = A' + Q$ and $B = B' + Q$. Then A and B are ideals which properly contain Q and $AB = (A' + Q)(B' + Q) \subseteq A'B' + Q = Q$.

(b) Suppose that Q is not prime and by (a) choose ideals A and B which properly contain Q and satisfy $AB \subseteq Q$. By the maximal nature of Q, there exist integers m and n with $A \supseteq I^m$ and $B \supseteq I^n$. Thus $Q \supseteq AB \supseteq I^m I^n = I^{m+n}$, contradicting the given property of Q. Consequently, Q must be prime.

(c) Let $\mathcal{S}$ be the set of ideals of R which do not contain any power of I. Since I is not nilpotent, it follows that $0 \in \mathcal{S}$ and hence $\mathcal{S}$ is nonempty. Since R satisfies the ascending chain condition on ideals, it satisfies the maximal condition. Thus $\mathcal{S}$ contains a maximal member, say Q. By (b), this ideal Q is prime and since $Q \in \mathcal{S}$ it follows that Q does not contain I.

3. (a) We know that $E = K[f(x)]$ is the splitting field over K of a separable polynomial $f(x) \in K[x]$. Furthermore, $F = K[g(x)]$. Thus clearly $L = EF = K[f(x)g(x)]$ and note that every irreducible factor of $f(x)g(x) \in K[x]$ divides $f(x)$ or $g(x)$. It follows that $f(x)g(x)$ is also separable and, consequently, L is Galois over K.

(b) Let $T = \mathrm{Gal}(L/K)$. Since E/K is Galois, it follows that every automorphism in T must stabilize E. Thus, by restriction, we obtain a group homomorphism $\theta\colon T = \mathrm{Gal}(L/K) \to \mathrm{Gal}(E/K) = G$. Similarly, restriction yields a homomorphism $\phi\colon T = \mathrm{Gal}(L/K) \to \mathrm{Gal}(F/K) = H$. By combining these we obtain a map $(\theta \times \phi)\colon T \to G \times H$ given by $(\theta \times \phi)(t) = (\theta(t), \phi(t)) \in G \times H$ for all t in T. Now, if t is in the kernel of this combined map, then $\theta(t) = 1$, so t fixes E elementwise. Similarly t fixes F elementwise. Thus t fixes $EF = L$ elementwise and hence $t = 1$. Thus $\ker(\theta \times \phi) = \langle 1 \rangle$ so $\theta \times \phi$ is a one-to-one map and T is isomorphic to its image, a subgroup of $G \times H$.

4. (a) The characteristic polynomial of the matrix $A(x) = \begin{bmatrix} 1 & -2 \\ 8 & x \end{bmatrix}$ in the variable λ is $\phi(\lambda) = \det(\lambda I - A(x)) = \lambda^2 - (x+1)\lambda + (x+16)$. Since the complex numbers are algebraically closed this polynomial factors into linear factors and we know that $A(x)$ will be diagonalizable if ϕ has distinct roots. So we need only be concerned with the possibility that the two roots are equal. In this case, the discriminant is 0 so $(x+1)^2 = 4(x+16)$. Hence $0 = x^2 - 2x - 63 = (x-9)(x+7)$ and $x = 9$ or -7. In the latter two cases, $A(x)$ has all eigenvalues equal. In particular, if it were diagonalizable, it would be similar to a scalar matrix and hence be scalar, a contradiction. Thus $x = 9, -7$ are really exceptions.

(b) Notice that $J^2 = nJ$, so J satisfies the polynomial $\lambda^2 - n\lambda = \lambda(\lambda - n)$ which has the distinct roots 0 and n. This implies that J is diagonalizable. Indeed, it will be similar to a diagonal matrix D having diagonal entries equal to 0 or n only, say a entries are 0 and b entries equal n. Of course $a + b = n$. Furthermore, since similar matrices have the same trace, we get $n = \operatorname{tr} J = \operatorname{tr} D = a0 + bn$. Thus $b = 1$, $a = n - 1$ and J is similar to $D = \operatorname{diag}(n, 0, 0, \ldots, 0)$.

Aternately, we could argue that $\operatorname{rank} J = 1$, so $\operatorname{rank} D = 1$ and hence D can have only one nonzero diagonal entry.

5. (a) Let $\tilde{F}$ be the algebraic closure of F and let $\alpha \in \tilde{F}$ be a root of $f(x)$ and $\beta \in \tilde{F}$ be a root of $g(y)$. Consider the evaluation map $\theta\colon F[x,y] \to \tilde{F}$ determined by $x \mapsto \alpha$ and $y \mapsto \beta$. Then both $f(x)$ and $g(y)$ are in the kernel of θ, so $I \subseteq \ker\theta$. But $1 \notin \ker\theta$, so $1 \notin I$ and $I \neq F[x,y]$.

(b) Since $x = f(x) + \alpha$ and $y = g(y) + \beta$, it follows that any polynomial in x and y is a multiple of $f(x)$ plus a multiple of $g(y)$ plus an element of F. Thus $F[x,y] = I + F$ and by (a) we know that $I \cap F = 0$. In other words, $F[x,y]/I \cong F$ is a field and hence I is a maximal ideal.

An alternate argument is as follows. Let $\theta\colon F[x,y] \to F$ be the evaluation map determined by $x \mapsto \alpha$, $y \mapsto \beta$, and let $J = \ker\theta$. Then $F[x,y]/J \cong F$, a field, so J is a maximal ideal and clearly $J \supseteq I$. On the other hand, note that $x \equiv \alpha \bmod I$ and $y \equiv \beta \bmod I$. Thus if $h(x,y) \in F[x,y]$, then $h(x,y) \equiv h(\alpha,\beta) \bmod I$. In particular, if $h(x,y) \in J$, then $h(x,y) \equiv h(\alpha,\beta) = 0 \bmod I$, so $h(x,y) \in I$. Thus $I \supseteq J$, and $I = J$ is maximal.