Answers to the Algebra Qualifying Exam
January 2000

1. (a) $n_{11}(H) \mid 5^3$ and $n_{11}(H) \equiv 1 \bmod 11$, so $n_{11}(H) = 1$.

(b) $n_5(G) \mid 2^4{\cdot}11$, $n_5(G) < 16$ and $n_5(G) \equiv 1 \bmod 5$, so $n_5(G) = 1$ or 11. If $n_5(G) = 1$, then G has a normal Sylow 5-subgroup. If $n_5(G) = 11$, then N, the normalizer of a Sylow 5-subgroup of G has index 11 in G. Now G permutes the right cosets of N by right multiplication, so we obtain a homomorphism $\theta\colon G \to \mathrm{Sym}_{11}$ with $K = \ker(\theta) \subseteq N$. Note that $|\mathrm{Sym}_{11}| = 11!$ is not divisible by 5^3. Thus $|K|$ is divisible by 5 and K is the required proper normal subgroup of G.

(c) If $n_5(G) = 16$, let H be the normalizer of a Sylow 5-subgroup of G. Then we have $|G : H| = 16$, so $|H| = 5^3{\cdot}11$. By part (a), H has a normal Sylow 11-subgroup Q. Note that Q is a Sylow 11-subgroup of G and that $\mathbb{N}_G(Q) \supseteq H$. Thus $n_{11}(G) = |G : \mathbb{N}_G(Q)|$ divides $|G : H| = 16$. But $n_{11}(G) \equiv 1 \bmod 11$, so $n_{11}(G) = 1$.

2. (a) We know that $MR = M \neq 0$ and hence, since $R = \sum_i I_i$, there exists k with $MI_k \neq 0$. Furthermore, since $I_k \triangleleft R$, MI_k is a nonzero submodule of the simple module M and hence $MI_k = M$. Finally, if $i \neq k$, then $I_kI_i \subseteq I_k \cap I_i = 0$. Thus we have $MI_i = MI_kI_i = M0 = 0$ and k is unique.

(b) Let $\theta\colon I_i \to I_j$ be a right module homomorphism with $i \neq j$. If $k \neq j$, then since $\theta(I_i) \subseteq I_j$, we have $\theta(I_i)I_k = 0$. If $k \neq i$, then $I_iI_k = 0$, so $\theta(I_i)I_k = \theta(I_iI_k) = \theta(0) = 0$. Thus $\theta(I_i)I_k = 0$ for all k. Hence $\theta(I_i)R = \sum_k \theta(I_i)I_k = 0$. But $1 \in R$, so $\theta(I_i) = 0$ and θ is the zero map.

3. (a) If E is a 2-tower over K, then $|E : K| = \prod |E_{i+1} : E_i| = 2^n$. Conversely, suppose $|E : K|$ is a power of 2, say 2^n. Now $L \supseteq E \supseteq K$, so under the Galois correspondence, $E = L^H$ where H is some subgroup of G. But G is abelian, so $H \triangleleft G$ and hence E is Galois over K with Galois group $A = G/H$, an abelian group of order 2^n. Clearly A has a composition series $A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n = 1$ with $|A_i : A_{i+1}| = 2$. If we set $E_i = E^{A_i}$, then $K = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$ with $|E_{i+1} : E_i| = |A_i : A_{i+1}| = 2$. Thus E is a 2-tower over K.

(b) As an example, let $G = \mathrm{Alt}_4$ act on the rational function field $L = F(x_1, x_2, x_3, x_4)$ by permuting the four variables. Let $K = L^G$ so that L/K is Galois with group G. Set $H = \mathrm{Alt}_3 \subseteq \mathrm{Alt}_4 = G$. Thus $|G : H| = 4$ and if $E = L^H$, then $|E : K| = |G : H| = 4 = 2^2$. If E/K is a 2-tower, then $E = E_2 > E_1 > E_0 = K$ and $E_1 = L^T$ for some subgroup T of G with $G > T > H$. But H is a maximal subgroup of G, so this is impossible.

4. (a) If B is the Jordan form of A, then B also has rank 1. Thus B must have one Jordan block

$$J = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

of rank 1 and all the other blocks are 0. Furthermore, if $\lambda \neq 0$, then $\operatorname{rank} J = 1$ implies that $J = (\lambda)$, while if $\lambda = 0$ then $\operatorname{rank} J = 1$ implies that $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus $B = \operatorname{diag}(\lambda, 0, \ldots, 0)$ if $\lambda \neq 0$ or $B = e_{1,2}$, the matrix unit, when $\lambda = 0$. In either case, note that $\lambda = \operatorname{trace}(B) = \operatorname{trace}(A)$.

(b) In the first case, it is clear that $\operatorname{char}(A) = \operatorname{char}(B) = x^{n-1}(x - \lambda)$ and $\min(A) = \min(B) = x(x - \lambda)$ since $\lambda \neq 0$. In the second case, $\operatorname{char}(A) = \operatorname{char}(B) = x^n$ and $\min(A) = \min(B) = x^2$.

5. (a) Suppose by way of contradiction that I is generated by the finitely many polynomials $a_1, a_2, \ldots, a_k \in I = xF[x, y]$. Let n be an integer strictly larger than the y-degrees of these k polynomials. By assumption, $xy^n \in I = a_1S + a_2S + \cdots + a_kS$ so, since $S = F + I$, we have

$$xy^n = a_1(f_1 + b_1) + a_2(f_2 + b_2) + \cdots + a_k(f_k + b_k)$$

where $f_1, f_2, \ldots, f_k \in F$ and $b_1, b_2, \ldots, b_k \in I = xF[x, y]$. Note that $a_ib_i \in I^2 = x^2F[x, y]$, so that all the monomials in these products are divisible by x^2. Furthermore, $a_1f_1 + a_2f_2 + \cdots + a_kf_k$ has y-degree strictly less than n. Thus xy^n cannot possibly be written as in the displayed equation and this is the required contradiction.

(b) We conclude from the above that S is not Noetherian. On the other hand, $R = F[x, y]$ is Noetherian by the Hilbert basis theorem. Let $J_1 < J_2 < J_3 \cdots$ be a strictly ascending chain of ideals of S. If infinitely many of these are ideals of R, then by deleting the remaining terms, we obtain a strictly increasing ascending chain of ideals of R, a contradiction since R is Noetherian. Thus at most finitely many of the J_i can be ideals of R and hence infinitely many of the J_i are ideals of S which are not ideals of R.