Answers to the Algebra Qualifying Exam
January 2001

1. (a) Let $H = \langle X, Y\rangle$ be the subgroup of G generated by X and Y. Certainly X and Y form a weird pair for H. We claim that X and Y are normal in H. If $y \in Y$, then X^y is a subgroup of H with $|X^y| = |X|$. Thus, by assumption, $X^y = X$ or Y. But $X^y = Y$ implies that $X = Y^{y^{-1}} = Y$, a contradiction. Thus $X^y = X$ and $y \in \mathbb{N}_H(X)$. It follows that $\mathbb{N}_H(X) \supseteq \langle X, Y\rangle = H$, so $X \triangleleft H$ and similarly $Y \triangleleft H$.

(b) Suppose, by way of contradiction, that $A \times 1$ and $1 \times B$ form a weird pair for $G = A \times B$. Then certainly $|A| = |A \times 1| = |1 \times B| = |B|$, and these orders are not equal to 1. Since A is solvable, $A' \neq A$ and thus A has a normal subgroup N of index p for some prime p. Then p divides $|A| = |B|$, so B has a subgroup P of order p. The combined map $\theta\colon A \to A/N \cong P \to B$ is then a nontrivial homomorphism from A to B. Let $C = \{(a, \theta(a)) \mid a \in A\} \subseteq A \times B = G$. Then it is easy to see that C is a subgroup of G different from $A \times 1$ and $1 \times B$. Furthermore, $C \cong A$ via the projection to the first coordinate. Thus $|C| = |A \times 1| = |1 \times B|$, contradicting the definition of weird pair.

(c) Let G be solvable and suppose, by way of contradiction, that X and Y form a weird pair for G. By part (a), we can assume that X and Y are normal in G and that $G = XY$. If $N = X \cap Y$, then $N \triangleleft G$ and it is easy to see that $A = X/N$ and $B = Y/N$ form a weird pair for G/N. Furthermore, $A, B \triangleleft G/N$, $A \cap B = 1$ and $G/N = AB$. Thus G/N is the internal direct product of A and B, and since A is solvable, this contradicts the result of part (b).

2. (a) By assumption, V has a composition series $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = V$ of length n. Let $W_i = V_i/V_{i-1}$ for $i = 1, \ldots, n$, so that each W_i is a simple R-module. Thus $P_i = \{r \in R \mid W_i r = 0\}$ is a primitive ideal of R and we set $I = \bigcap_{i=1}^n P_i$. By definition of $J = \mathrm{Jrad}(R)$, we know that $I \supseteq J$. Since $I \subseteq P_i$, we have $W_i I = 0$ and hence $V_i I \subseteq V_{i-1}$. It follows by induction that $V_i I^i \subseteq V_0 = 0$, so $VI^n = V_n I^n = 0$. Since R acts faithfully on V, this yields $I^n = 0$. But $\mathrm{Jrad}(R)$ contains all nilpotent ideals of R, so $J \supseteq I$ and we conclude that $J = I$. In particular, $J = I$ is an intersection of the n primitive ideals P_i, and $J^n = I^n = 0$.

(b) Let K be a field and let R be the subring of the 2×2 matrix ring $\mathrm{M}_2(K)$ with $R = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$. Then R acts faithfully (on the right) on $V = (K, K) = K^2$, the K-vector space of 1×2 row vectors. If $V_1 = (0, K) \subseteq V$, then V_1 is an R-submodule of V and $0 \subseteq V_1 \subseteq V$ is a composition series of length 2. This follows since $R \supseteq K$ implies that any R-module W with $\dim_K W = 1$ must be irreducible. Finally, $\mathrm{Jrad}(R) \neq 0$ since $I = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ is a nonzero nilpotent ideal of the ring R.

3. (a) Let the roots of $f(x)$ be $\alpha, 2\alpha, \beta_1, \ldots, \beta_k$. Since f is monic, each of these elements is an algebraic integer, and their product is $\pm f(0)$, since $f(0)$ is the constant term of the polynomial. In particular, if $f(0) = 1$, then $\alpha \cdot 2\alpha \cdot \prod_1^k \beta_i = \pm 1$, so we have

$1/2 = \pm\alpha^2 \cdot \prod_1^k \beta_i$. Thus 1/2 is a noninteger rational number that is an algebraic integer. This is a contradiction since $\mathbb{Z}$ is a UFD and hence integrally closed. The latter means that the only elements of the rationals $\mathbb{Q}$ integral over $\mathbb{Z}$ are the elements of $\mathbb{Z}$ itself.

(b) Let K be the splitting field of $f(x)$ over $\mathbb{Q}$ and let G be the Galois group of $K/\mathbb{Q}$. Since f is irreducible, we know that G is finite and that G permutes the roots of f transitively. In particular, there exists $\sigma \in G$ with $\sigma(\alpha) = 2\alpha$. Then, by induction, $\sigma^n(\alpha) = 2^n\alpha$. But σ has finite order, say $m \geq 1$, so $2^m\alpha = \sigma^m(\alpha) = \mathrm{Id}(\alpha) = \alpha$. Thus $(2^m - 1)\alpha = 0$ and since K has characteristic 0, we conclude that $\alpha = 0$.

4. (a) Since the complex numbers are algebraically closed, all eigenvalues of complex matrices are contained in the complex numbers. Now we know that $\langle v, w\rangle = v^*w$ defines an Hermitian inner product on the space of complex $n \times 1$ column vectors. In particular, $\langle v, v\rangle$ is always real and nonnegative. Now let λ be an eigenvalue of A^*A with corresponding eigenvector v. Then $A^*Av = \lambda v$, so $v^*A^*Av = \lambda v^*v$. Note that $v^*A^*Av = \langle Av, Av\rangle \geq 0$ and that $v^*v = \langle v, v\rangle > 0$ since $v \neq 0$. Thus $\lambda = \langle Av, Av\rangle/\langle v, v\rangle$ is real and nonnegative.

(b) If λ is an eigenvalue of $I + A^*A$ with corresponding eigenvector v, then $v + A^*Av = (I + A^*A)v = \lambda v$, so $A^*Av = (\lambda - 1)v$. Hence $\lambda - 1$ is an eigenvalue of A^*A, so (a) implies that $\lambda - 1 \geq 0$. Thus λ is real and positive. Since $\det(I + A^*A)$ is the product of the eigenvalues of $I + A^*A$, we conclude that $\det(I + A^*A)$ is real and positive.

5. (a) Since $S \neq 0$, we know that $W = \{v \in V \mid vS = 0\}$ is a subspace of V different from V. Furthermore, $S \in F[T]$ implies that S and T commute. Thus $vS = 0$ yields $(vT)S = (vS)T = 0T = 0$, so $WT \subseteq W$ and the hypothesis implies that $W = 0$.

(b) $F[T]$ is certainly a commutative ring acting faithfully on V. If $0 \neq S \in F[T]$, then we know from part (a) that S is a nonsingular linear transformation and hence that S^{-1} exists. Since S satisfies its characteristic polynomial, we have $S^r + a_1S^{r-1} + \cdots + a_rI = 0$ where $a_i \in F$ and $r = \dim_F V$. Furthermore, $a_r \neq 0$ since S is nonsingular. We can now multiply the polynomial equation by S^{-1} and a_r^{-1} to obtain

$$S^{-1} = -(a_r^{-1}S^{r-1} + a_r^{-1}a_1S^{r-2} + \cdots + a_r^{-1}a_{r-1}I) \in F[T].$$

Hence $F[T]$ is a field.

(c) We know that $F[T]$ acts faithfully on V, so V is a vector space over this larger field. Since any subspace of V is T-stable, the hypothesis implies that V must be 1-dimensional over $F[T]$. Thus if $0 \neq v \in V$, then $V = vF[T] \cong F[T]$ as right $F[T]$-vector spaces and hence as right F-vector spaces. By definition of the degree of a field extension, we conclude that $\dim_F V = \dim_F F[T] = |F[T] : F|$.