Answers to the Algebra Qualifying Exam
January 2002

1. a. $G/N = NH/N \cong H/(H \cap N) = H/1 \cong H$. Thus every complement of N is isomorphic to G/N, and hence all complements are isomorphic.

b. We apply the Sylow theorems. Let H be a p-group and let S be a Sylow p-subgroup of G containing H. If P is any Sylow p-subgroup of G, then $P = S^g = g^{-1}Sg$ for some $g \in G$, so $P = S^g \supseteq H^g$. We claim that H^g is also a complement for N. Indeed, $G = NH$, so $G = G^g = N^g H^g = NH^g$ since $N \triangleleft G$. Also $1 = 1^g = (N \cap H)^g = N^g \cap H^g = N \cap H^g$.

c. Since $N \triangleleft G$, we know that G acts as automorphisms of N via conjugation. Specifically, there is a homomorphism $\theta\colon G \to \mathrm{Aut}(N)$ with kernel $C = \mathbb{C}_G(N) \triangleleft G$. Under this map θ, the image of N is $\mathrm{Inn}(N)$, the group of inner automorphisms. By assumption, $\mathrm{Inn}(N) = \mathrm{Aut}(N)$, so $\theta(N) = \mathrm{Aut}(N)$. It follows that NC contains $\ker\theta = C$ and maps onto $\mathrm{Aut}(N) = \theta(G)$, so $NC = G$. Also $C \cap N = \mathbb{C}_G(N) \cap N = \mathbb{Z}(N) = 1$, by assumption. In other words, $C = \mathbb{C}_G(N)$ is a normal complement for N in G.

Finally, let H be any normal complement for N. Since N and H are disjoint normal subgroups, they commute and hence $H \subseteq C = \mathbb{C}_G(N)$. Thus since C is disjoint from N and since $H \subseteq C$, we obtain $C = NH \cap C = (N \cap C)H = H$, and $H = \mathbb{C}_G(N)$ is unique.

2. a. Let F be the field of fractions of S. We will work in $\overline{F}$, the algebraic closure of F. Note that $\overline{F}$ contains K, the field of fractions of R, and also $\overline{K}$, the algebraic closure of K. Let $I = \{f(x) \in R[x] \mid f(s) = 0\}$ and let $I' = \{f(x) \in K[x] \mid f(s) = 0\}$. Then $I \triangleleft R[x]$, $I' \triangleleft K[x]$ and $I \subseteq I'$. Since s is integral over R, we know that I contains a monic polynomial $h(x) \in R[x]$.

Now $K[x]$ is a principal ideal domain, so I' is the principal ideal generated by a monic polynomial $g(x) \in K[x]$. In particular, since $I \subseteq I'$, every element of I is a $K[x]$-multiple of $g(x)$. Thus $g(x)$ divides $h(x)$, and hence all roots of $g(x)$ are roots of $h(x)$ and are consequently integral over R. Furthermore, since the coefficients of $g(x)$ are $\pm$ sums of products of the roots, we see that all coefficients of $g(x) \in K[x]$ are elements of K that are integral over R. But R is integrally closed in K, by assumption, so $g(x) \in R[x] \cap I' = I$. We claim that I is the principal ideal generated by $g(x)$. To this end, let $f(x) \in I$. Then, since $g(x)$ is monic, long division implies that $f(x) = q(x)g(x) + r(x)$, where $q(x), r(x) \in R[x]$ and $\deg r(x) < \deg g(x)$. But then $r(x) \in I \subseteq I'$, so $g(x)$ divides $r(x)$, and hence $r(x) = 0$ by degree considerations. In other words, $f(x) \in g(x)R[x]$, as required.

b. Since I is a prime ideal, $S = R[x]/I$ is an integral domain. Since $I \cap R = 0$, R embeds isomorphically into S and, without loss of generality, we can assume that $R \subseteq S$. With this assumption, we see that the map $\theta\colon R[x] \to S = R[x]/I$ is given by $\theta(f(x)) = f(s)$ where $s = x + I \in S$. Thus, since $I = \ker\theta$, it follows that $I = \{f(x) \in R[x] \mid f(s) = 0\}$. Finally, since I contains a monic polynomial, s is integral over R, and hence I is a principal ideal by the result of part (a).

3. a. Note that $F = K(t)$ is the field of fractions of the unique factorization domain $K[t]$ and that t is a prime element of $R = K[t]$. Also observe that t divides all coefficients of $f(x) \in R[x]$ other than the leading term, and that t^2 does not divide the constant term of this polynomial. Thus, by Eisenstein's criterion, $f(x)$ is irreducible in $F[x]$.

b. We are given $E = F[s]$, where $s^p = t$. Thus in $E[x]$ we have $f(x) = x^{2p} - tx^p + t = x^{2p} - s^p x^p + s^p = (x^2 - sx + s)^p$. In particular, the splitting field L of $f(x)$ over E is the splitting field of $x^2 - sx + s$ over E. Since the two roots of this quadratic sum to $s \in E$, any single root generates L over E and hence $|L : E| \le 2$.

c. Let α be any root of $f(x)$ in the algebraic closure of E. Then $\alpha \in L$ and, since $f(x)$ is irreducible over F, we have $|L : F| \ge |F[\alpha] : F| = \deg f(x) = 2p$. On the other hand, $E = F[s]$ with $s^p = t$, so $|E : F| \le p$. Also, by part (b), $|L : E| \le 2$. Thus $|L : F| = |L : E||E : F| \le 2p$. This is the reverse inequality, so we conclude that equalities occur throughout. In particular, $|L : F| = 2p$ and $L = F[\alpha]$.

4. a. $v_0K[T]$ is a T-invariant subspace of V containing v_0. Hence, by assumption, $v_0K[T] = V$. Consider the linear transformation $\alpha\colon K[T] \to V$ given by $f(T) \mapsto v_0 f(T)$. Since $\alpha(K[T]) = v_0K[T] = V$, it follows that α is onto and therefore $\dim_K K[T] \ge \dim_K V = n$.

b. Let $W = \{v \in V \mid vS = 0\}$ so that W is a K-subspace of V which contains v_0, by assumption. Furthermore, since $S \in \mathcal{C}$, $(WT)S = (WS)T = 0T = 0$, and hence $WT \subseteq W$. In other words, W is T-invariant and consequently $W = V$, by assumption. Thus S annihilates all of V, and $S = 0$ as required. Now consider the linear transformation $\beta\colon \mathcal{C} \to V$ given by $C \mapsto v_0C$. Then the above implies that $\ker\beta = \{S \in \mathcal{C} \mid v_0S = 0\} = 0$. Thus β is one-to-one and therefore $\dim_K \mathcal{C} \le \dim_K V = n$.

c. Note that any polynomial in T commutes with T, so $K[T] \subseteq \mathcal{C}$ and $\dim_K K[T] \le \dim_K \mathcal{C}$. But we have shown in (a) and (b) that $\dim_K K[T] \ge n \ge \dim_K \mathcal{C}$. Thus we must have equality throughout, so $\dim_K K[T] = n = \dim_K \mathcal{C}$ and $K[T] = \mathcal{C}$. Finally consider the algebra homomorphism $\gamma\colon K[x] \to K[T]$ given by $f(x) \mapsto f(T)$. Since the polynomial ring $K[x]$ is a principal ideal domain, the kernel of γ is generated by $g(x)$, the minimal polynomial of T. Now we know that $K[x]/(g(x))$ is K-vector space isomorphic to the set of all polynomials of degree less than that of $g(x)$. Thus $\deg g(x) = \dim_K K[x]/(g(x)) = \dim_K K[T] = n$.

5. Let $\operatorname{len}(V)$ denote the composition length of V. By the Jordan-Hölder theorem, $\operatorname{len}(V)$ is well defined. Furthermore, if W is a submodule of V, then $\operatorname{len}(W) + \operatorname{len}(V/W) = \operatorname{len}(V)$. Of course, $W = 0$ if and only if $\operatorname{len}(W) = 0$. Furthermore, $W = V$ if and only if $V/W = 0$ and hence if and only if $\operatorname{len}(W) = \operatorname{len}(V)$. Since V has a finite composition series, the R-submodules of V satisfy both the maximal and minimal conditions. In particular, every nonzero submodule of V contains a minimal submodule of V.

a. Let $\theta \in E$ and set $K = \ker\theta$ and $L = \operatorname{im}\theta$. These are R-submodules of V with $V \supseteq L \cong V/K$, so $\operatorname{len}(K) + \operatorname{len}(L) = \operatorname{len}(V)$. Now θ is one-to-one if and only if $K = 0$ and hence if and only if $\operatorname{len}(L) = \operatorname{len}(V)$. As we observed, the latter occurs if and only if $L = V$ and hence if and only if θ is onto. Thus θ is one-to-one if and only if it is onto and hence if and only if it is invertible. Of course, if θ is invertible as a function, then $\theta^{-1} \in E$.

b. Let U be the unique minimal submodule of V. Then $\theta \in E$ is not one-to-one if and only if $\ker\theta \ne 0$ and hence if and only if $\ker\theta \supseteq U$. Set $I = \{\theta \in E \mid \ker\theta \supseteq U\}$. By (a) and the above observation, I is precisely the set of noninvertible elements of E. If $\alpha, \beta \in I$, then $\alpha(U) = \beta(U) = 0$, so $(\alpha + \beta)(U) = 0$ and $\alpha + \beta \in I$. Furthermore, if $\gamma \in E$, then neither $\gamma\alpha$ nor $\alpha\gamma$ can be invertible since α is not invertible. Thus $\gamma\alpha, \alpha\gamma \in I$ and I is an ideal of E.

c. Suppose $0 \ne \theta \in E$ is not invertible. Then we know that θ is not one-to-one, so $K = \ker\theta \supseteq U$. Since $\theta \ne 0$, it follows that $0 \ne \theta(V) \subseteq V$ and hence $V/K \cong \theta(V) \supseteq U$. But then both V/K and K have U as a composition factor, so U occurs with multiplicity ≥ 2 in a composition series for V. This contradicts our assumption, so we conclude that every nonzero element of E is invertible. Consequently, E is a division ring.