1. (a) We have $K \subseteq AB$, so $A \subseteq KA \subseteq AB$ and it follows from the modular law that $KA = A(KA \cap B)$. Since $B$ is minimal normal, we have $KA \cap B = 1$ or $B$. In the former case, $KA = A(KA \cap B) = A$, so $K \subseteq A$ and hence $K = A$, contradiction. Thus $KA \cap B = B$ and $KA = A(KA \cap B) = AB$. Similarly, $KB = AB$.

(b) Since $A$, $B$ and $K$ are minimal normal subgroups of $G$ with $K \neq A$ and $K \neq B$, we have $K \cap A = K \cap B = 1$. Furthermore, $K \subseteq AB$ and $K \neq A$, so $A \neq B$ and $A \cap B = 1$. It follows that $K \times A \cong KA = AB \cong A \times B$, so $K \cong KA/A = AB/A \cong B$. Similarly, we have $K \cong B$.

(c) Since $A \cap B = A \cap K = 1$ and since these are normal subgroups of $G$, the commutators satisfy $[A, B] \subseteq A \cap B = 1$ and $[A, K] \subseteq A \cap K = 1$. In particular, $A$ centralizes both $K$ and $B$. But $A \subseteq AB = KB$, so $A$ centralizes $A$ and hence $A$ is abelian. Since $A \cong B$, we see that $B$ is abelian and hence so is $AB \cong A \times B$.

2. (a) Note that $x^3, x^4 \in R$, so $x = x^4/x^3$ belongs to the field of fractions $F$ of $R$. Thus $F \supseteq \mathbb{Z}[x]$ and hence $F \supseteq \mathbb{Q}(x)$, the field of fractions of $\mathbb{Z}[x]$. On the other hand, $R \subseteq \mathbb{Q}(x)$, so clearly $F \subseteq \mathbb{Q}(x)$ and we have equality.

(b) Let $S$ be the integral closure of $R$ in $F = \mathbb{Q}(x)$. Since $x^3 \in R$, we see that $x$ is integral over $R$ and hence $x \in S$. Thus $\mathbb{Q}(x) \supseteq S \supseteq \mathbb{Z}[x] \supseteq R$. Now $\mathbb{Z}$ is a p.i.d., so $\mathbb{Z}[x]$ is a u.f.d. and hence $\mathbb{Z}[x]$ is integrally closed in its field of fractions. Since any element of $S$ is integral over $R$, it is integral over $\mathbb{Z}[x]$ and hence belongs to $\mathbb{Z}[x]$. Thus $S = \mathbb{Z}[x]$.

(c) Suppose $R$ is generated by 1 and $g(x)$. Then certainly $g(x) \notin \mathbb{Z}$, so $\text{deg} g(x) \geq 1$. Let $\mathbb{Z}[t]$ be the polynomial ring in the indeterminant $t$ and consider the homomorphism $\theta: \mathbb{Z}[t] \to R$ given by $t \mapsto g(x)$. We note that $\theta$ is one-to-one. Otherwise, if $f(t) = \sum_{i=0}^{n} a_i t^i$ is a nontrivial polynomial in the kernel of $\theta$, with $a_n \neq 0$, then $0 = \theta(f) = \sum_{i=0}^{n} a_i g(x)^i$. But $\text{deg} g(x)^i = i \text{deg} g(x)$, so $\sum_{i=0}^{n} a_i g(x)^i$ has degree $n \text{deg} g(x)$, a contradiction. Thus $\theta$ is one-to-one and it is onto since 1 and $g(x)$ generate $R$. Thus $R \cong \mathbb{Z}[t]$. But then $R$ is a u.f.d., so it is integrally closed in its field of fractions and this contradicts the result of part (b) that $S \neq R$.

Alternately, we could note as above that any polynomial in $g(x)$ has degree divisible by $\text{deg} g(x)$. But $x^3, x^4 \in R$, so $\text{deg} g(x)$ divides 3 and 4. Hence $\text{deg} g(x) = 1$ and this contradicts the definition of $R$.

3. (a) Since the characteristic of $F$ does not divide $n$, we see that $f'(x) = nx^{n-1}$ has only 0 as a root. Thus $f(x)$ and its derivative $f'(x)$ have no roots in common, so we conclude that $f(x)$ has $n$ distinct roots, say $b = b_1, b_2, \ldots, b_n$. Now $(b_i/b)^n = a/a = 1$, so $b_i/b = \varepsilon_i$ is an $n$th root of unity. Indeed, in this way, we obtain $n$ distinct $n$th roots of unity in $E$, so $E$ contains a primitive $n$th root $\varepsilon$.

(b) Suppose $F$ contains $\varepsilon$, a primitive $n$th root of unity. Then $F$ contains all $n$th roots of unity. Now let $g(x)$ be any irreducible factor of $f(x)$ in $F[x]$. If $b_i$ is a root of $g(x)$, then $F[b_i]$ contains $b_i \varepsilon^j$ for all $j$ and hence it contains all roots of $f(x)$. Thus $F[b_i] = E$ and, since $g(x)$ is irreducible, $\text{deg} g(x) = |F[b_i] : F| = |E : F|$. We conclude that all these irreducible factors have the same degree namely $|E : F|$. Finally, factor $f(x)$ in $F[x]$ as
\[ f(x) = \prod_{k=1}^{m} f_k(x), \] with each \( f_k(x) \) irreducible. Then \( n = \deg f(x) = \sum_{k=1}^{m} \deg f_k(x) = m \cdot |E : F| \) and hence \( |E : F| \) divides \( n \).

(c) Assume now that \( n \) is a power of 2. We know that \( \varepsilon \in E \) so \( E \supseteq F[\varepsilon] \supseteq F \). Since \( f(x) \in F[\varepsilon][x] \), part (b) applied to \( E \supseteq F[\varepsilon] \) implies that \( |E : F[\varepsilon]| \) divides \( n \). Thus \( |E : F[\varepsilon]| \) is a power of 2. Since \( |E : F| = |E : F[\varepsilon]| \cdot |F[\varepsilon] : F| \), it now suffices to show that \( |F[\varepsilon] : F| \) is a power of 2. For this, we proceed by induction on \( n \). Let \( m = n/2 \) and let \( \delta \) be a primitive \( m \)th root of unity. Then \( F[\varepsilon] \supseteq F[\delta] \supseteq F \) and \( |F[\delta] : F| \) is a power of 2 by induction. Since \( \varepsilon^2 \in F[\delta] \), we see that \( |F[\varepsilon] : F[\delta]| = 1 \) or 2. The result now follows since \( |F[\varepsilon] : F| = |F[\varepsilon] : F[\delta]| \cdot |F[\delta] : F| \).

4. (a) The eigenvalues of \( A \) are the real roots of its characteristic polynomial. If \( \dim V = m \) is odd, then the characteristic polynomial \( f(x) \in \mathbb{R}[x] \) of \( A \) is a monic polynomial of odd degree. Thus, we need only show that a real (monic) polynomial of odd degree has a real root. One way is to use Gauss’ theorem which implies that \( f(x) \) is a product of linear and quadratic factors. Since \( \deg f(x) \) is odd, the factors cannot all be quadratic. So there is a linear factor and hence a real root. Alternately, we can apply the intermediate value theorem to the continuous real-valued function \( f(x) \). Since the degree is odd, we have \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = \infty \). Thus, by the intermediate value theorem, \( f(x) \) takes on all real values and in particular \( f(x) \) takes on the value 0.

(b) We proceed by induction on \( \dim V \). Suppose first that \( V \) has a proper subspace \( W \) that is invariant under all the operators \( A_i \). Since the eigenvalues of \( A_i \) on \( W \) and \( V/W \) are necessarily eigenvalues of \( A_i \) on \( V \), it follows easily that the hypotheses are satisfied when the \( A_i \) are restricted to \( W \) or to \( V/W \). Hence, by induction, \( \dim W \) and \( \dim V/W \) are both even. Since \( \dim V = \dim W + \dim V/W \), the result follows in this case.

Now suppose that no such \( W \neq 0 \) or \( V \) exists and assume by way of contradiction that \( \dim V \) is odd. By part (a), each \( A_i \) has a real eigenvalue, say \( r_i \), and hence a nonzero eigenspace \( W_i = \{ v \in V \mid vA_i = r_i v \} \). But each \( A_j \) commutes with \( A_i \) and hence \( W_i \) is stable under each \( A_j \). By assumption, \( W_i = V \) and hence \( A_i \) acts like \( r_i I \) on \( V \). Thus \( -I = \sum_i A_i = (\sum_i r_i)I \), so \( \sum_i r_i = -1 \). But, by assumption, each \( r_i \) is a nonnegative real number, so we have a contradiction and hence \( \dim V \) is even in this case also.

5. (a) Let \( M \) be generated by its artinian submodules so that \( M = \sum_i A_i \) (a possibly infinite sum) where each \( A_i \) is artinian. If \( \overline{M} \) is a nonzero homomorphic image of \( M \), then \( \overline{M} = \sum_i \overline{A_i} \) where \( \overline{A_i} \) is the image of \( A_i \) in \( M \). But a homomorphic image of an artinian module is artinian, so each \( \overline{A_i} \) is artinian. Since \( \overline{M} \neq 0 \), some \( \overline{A_i} \) is not zero and this \( \overline{A_i} \) has a nonzero minimal submodule \( \overline{B} \). Then \( \overline{B} \) is a simple submodule of \( \overline{M} \) and hence \( M \) has property (*).

(b) Assume that \( M \) has property (*) and that it is noetherian. Consider the set \( \mathcal{A} \) of all artinian submodules of \( M \). Then \( 0 \in \mathcal{A} \), so \( \mathcal{A} \) is nonempty. By the noetherian property, \( \mathcal{A} \) has a maximal member \( A \). We claim that \( A = M \). If not, then \( \overline{M} = M/A \) is nonzero and hence, by property (*), \( \overline{M} \) has a nonzero simple submodule \( \overline{B} \), where \( B \subseteq M \) is the complete inverse image in \( M \) of \( \overline{B} \). Note that \( B \supseteq A \) and that \( B/A \cong \overline{B} \) is simple and hence artinian. Thus since \( A \) is artinian, we conclude that \( B \) is artinian. In other words, \( B \in \mathcal{A} \). But \( B \) is properly larger than \( A \) and this contradicts the maximality of \( A \in \mathcal{A} \). We conclude that \( M = A \) is artinian, as required.