Answers to the Algebra Qualifying Exam
January 2006

1. (a) We have $K \subseteq AB$, so $A \subseteq KA \subseteq AB$ and it follows from the modular law that $KA = A(KA \cap B)$. Since B is minimal normal, we have $KA \cap B = 1$ or B. In the former case, $KA = A(KA \cap B) = A$, so $K \subseteq A$ and hence $K = A$, contradiction. Thus $KA \cap B = B$ and $KA = A(KA \cap B) = AB$. Similarly, $KB = AB$.

(b) Since A, B and K are minimal normal subgroups of G with $K \neq A$ and $K \neq B$, we have $K \cap A = K \cap B = 1$. Furthermore, $K \subseteq AB$ and $K \neq A$, so $A \neq B$ and $A \cap B = 1$. It follows that $K \times A \cong KA = AB \cong A \times B$, so $K \cong KA/A = AB/A \cong B$. Similarly, we have $K \cong B$.

(c) Since $A \cap B = A \cap K = 1$ and since these are normal subgroups of G, the commutators satisfy $[A, B] \subseteq A \cap B = 1$ and $[A, K] \subseteq A \cap K = 1$. In particular, A centralizes both K and B. But $A \subseteq AB = KB$, so A centralizes A and hence A is abelian. Since $A \cong B$, we see that B is abelian and hence so is $AB \cong A \times B$.

2. (a) Note that $x^3, x^4 \in R$, so $x = x^4/x^3$ belongs to the field of fractions F of R. Thus $F \supseteq \mathbb{Z}[x]$ and hence $F \supseteq \mathbb{Q}(x)$, the field of fractions of $\mathbb{Z}[x]$. On the other hand, $R \subseteq \mathbb{Q}(x)$, so clearly $F \subseteq \mathbb{Q}(x)$ and we have equality.

(b) Let S be the integral closure of R in $F = \mathbb{Q}(x)$. Since $x^3 \in R$, we see that x is integral over R and hence $x \in S$. Thus $\mathbb{Q}(x) \supseteq S \supseteq \mathbb{Z}[x] \supseteq R$. Now $\mathbb{Z}$ is a p.i.d., so $\mathbb{Z}[x]$ is a u.f.d. and hence $\mathbb{Z}[x]$ is integrally closed in its field of fractions. Since any element of S is integral over R, it is integral over $\mathbb{Z}[x]$ and hence belongs to $\mathbb{Z}[x]$. Thus $S = \mathbb{Z}[x]$.

(c) Suppose R is generated by 1 and $g(x)$. Then certainly $g(x) \notin \mathbb{Z}$, so $\deg g(x) \geq 1$. Let $\mathbb{Z}[t]$ be the polynomial ring in the indeterminant t and consider the homomorphism $\theta \colon \mathbb{Z}[t] \to R$ given by $t \mapsto g(x)$. We note that θ is one-to-one. Otherwise, if $f(t) = \sum_{i=0}^{n} a_i t^i$ is a nontrivial polynomial in the kernel of θ, with $a_n \neq 0$, then $0 = \theta(f) = \sum_{i=0}^{n} a_i g(x)^i$. But $\deg g(x)^i = i \deg g(x)$, so $\sum_{i=0}^{n} a_i g(x)^i$ has degree $n \deg g(x)$, a contradiction. Thus θ is one-to-one and it is onto since 1 and $g(x)$ generate R. Thus $R \cong \mathbb{Z}[t]$. But then R is a u.f.d., so it is integrally closed in its field of fractions and this contradicts the result of part (b) that $S \neq R$.

Alternately, we could note as above that any polynomial in $g(x)$ has degree divisible by $\deg g(x)$. But $x^3, x^4 \in R$, so $\deg g(x)$ divides 3 and 4. Hence $\deg g(x) = 1$ and this contradicts the definition of R.

3. (a) Since the characteristic of F does not divide n, we see that $f'(x) = nx^{n-1}$ has only 0 as a root. Thus $f(x)$ and its derivative $f'(x)$ have no roots in common, so we conclude that $f(x)$ has n distinct roots, say $b = b_1, b_2, \ldots, b_n$. Now $(b_i/b)^n = a/a = 1$, so $b_i/b = \varepsilon_i$ is an nth root of unity. Indeed, in this way, we obtain n distinct nth roots of unity in E, so E contains a primitive nth root ε.

(b) Suppose F contains ε, a primitive nth root of unity. Then F contains all nth roots of unity. Now let $g(x)$ be any irreducible factor of $f(x)$ in $F[x]$. If b_i is a root of $g(x)$, then $F[b_i]$ contains $b_i \varepsilon^j$ for all j and hence it contains all roots of $f(x)$. Thus $F[b_i] = E$ and, since $g(x)$ is irreducible, $\deg g(x) = |F[b_i] : F| = |E : F|$. We conclude that all these irreducible factors have the same degree namely $|E : F|$. Finally, factor $f(x)$ in $F[x]$ as

$f(x) = \prod_{k=1}^{m} f_k(x)$, with each $f_k(x)$ irreducible. Then $n = \deg f(x) = \sum_{k=1}^{m} \deg f_k(x) = m{\cdot}|E : F|$ and hence $|E : F|$ divides n.

(c) Assume now that n is a power of 2. We know that $\varepsilon \in E$ so $E \supseteq F[\varepsilon] \supseteq F$. Since $f(x) \in F[\varepsilon][x]$, part (b) applied to $E \supseteq F[\varepsilon]$ implies that $|E : F[\varepsilon]|$ divides n. Thus $|E : F[\varepsilon]|$ is a power of 2. Since $|E : F| = |E : F[\varepsilon]|{\cdot}|F[\varepsilon] : F|$, it now suffices to show that $|F[\varepsilon] : F|$ is a power of 2. For this, we proceed by induction on n. Let $m = n/2$ and let δ be a primitive mth root of unity. Then $F[\varepsilon] \supseteq F[\delta] \supseteq F$ and $|F[\delta] : F|$ is a power of 2 by induction. Since $\varepsilon^2 \in F[\delta]$, we see that $|F[\varepsilon] : F[\delta]| = 1$ or 2. The result now follows since $|F[\varepsilon] : F| = |F[\varepsilon] : F[\delta]|{\cdot}|F[\delta] : F|$.

4. (a) The eigenvalues of A are the real roots of its characteristic polynomial. If $\dim V = m$ is odd, then the characteristic polynomial $f(x) \in \mathbb{R}[x]$ of A is a monic polynomial of odd degree. Thus, we need only show that a real (monic) polynomial of odd degree has a real root. One way is to use Gauss' theorem which implies that $f(x)$ is a product of linear and quadratic factors. Since $\deg f(x)$ is odd, the factors cannot all be quadratic. So there is a linear factor and hence a real root. Alternately, we can apply the intermediate value theorem to the continuous real-valued function $f(x)$. Since the degree is odd, we have $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. Thus, by the intermediate value theorem, $f(x)$ takes on all real values and in particular $f(x)$ takes on the value 0.

(b) We proceed by induction on $\dim V$. Suppose first that V has a proper subspace W that is invariant under all the operators A_i. Since the eigenvalues of A_i on W and on V/W are necessarily eigenvalues of A_i on V, it follows easily that the hypotheses are satisfied when the A_i are restricted to W or to V/W. Hence, by induction, $\dim W$ and $\dim V/W$ are both even. Since $\dim V = \dim W + \dim V/W$, the result follows in this case.

Now suppose that no such $W \neq 0$ or V exists and assume by way of contradiction that $\dim V$ is odd. By part (a), each A_i has a real eigenvalue, say r_i, and hence a nonzero eigenspace $W_i = \{v \in V \mid vA_i = r_i v\}$. But each A_j commutes with A_i and hence W_i is stable under each A_j. By assumption, $W_i = V$ and hence A_i acts like $r_i I$ on V. Thus $-I = \sum_i A_i = (\sum_i r_i)I$, so $\sum_i r_i = -1$. But, by assumption, each r_i is a nonnegative real number, so we have a contradiction and hence $\dim V$ is even in this case also.

5. (a) Let M be generated by its artinian submodules so that $M = \sum_i A_i$ (a possibly infinite sum) where each A_i is artinian. If $\overline{M}$ is a nonzero homomorphic image of M, then $\overline{M} = \sum_i \overline{A}_i$ where $\overline{A}_i$ is the image of A_i in $\overline{M}$. But a homomorphic image of an artinian module is artinian, so each $\overline{A}_i$ is artinian. Since $\overline{M} \neq 0$, some $\overline{A}_i$ is not zero and this $\overline{A}_i$ has a nonzero minimal submodule $\overline{B}$. Then $\overline{B}$ is a simple submodule of $\overline{M}$ and hence M has property $(*)$.

(b) Assume that M has property $(*)$ and that it is noetherian. Consider the set $\mathcal{A}$ of all artinian submodules of M. Then $0 \in \mathcal{A}$, so $\mathcal{A}$ is nonempty. By the noetherian property, $\mathcal{A}$ has a maximal member A. We claim that $A = M$. If not, then $\overline{M} = M/A$ is nonzero and hence, by property $(*)$, $\overline{M}$ has a nonzero simple submodule $\overline{B}$, where $B \subseteq M$ is the complete inverse image in M of $\overline{B}$. Note that $B \supseteq A$ and that $B/A \cong \overline{B}$ is simple and hence artinian. Thus since A is artinian, we conclude that B is artinian. In other words, $B \in \mathcal{A}$. But B is properly larger than A and this contradicts the maximality of $A \in \mathcal{A}$. We conclude that $M = A$ is artinian, as required.