Answers to the Algebra Qualifying Exam
January 1994

1. (i) Let G be a perfect group, let S be a nonidentity solvable group and suppose $\phi: G \to S$ is an epimorphism. Since S is nonidentity and solvable, its commutator quotient S/S' is a nonidentity abelian group. Thus the composite map $G \to S \to S/S'$ yields a homomorphism of G onto a nontrivial abelian group and this contradicts the fact that G is perfect.

(ii) Now let $H \triangleleft G$ with G/H perfect. Suppose $\theta: G \to S$ is a homomorphism from G to a solvable group S and let $N = \ker\theta$. Then G/N is solvable since it is isomorphic to a subgroup of S. Notice that $NH \triangleleft G$ and that G/NH is a homomorphic image of G/N. Thus G/NH is also solvable. On the other hand, G/NH is a homomorphic image of the perfect group G/H. Thus, by (i) above, $G/NH = 1$ and $G = NH$. Since $\theta(N) = 1$, this yields $\theta(G) = \theta(NH) = \theta(N)\theta(H) = \theta(H)$.

2. (i) Let $W \subseteq V$ and let X be a simple submodule of W. By assumption, X is a direct summand of V, say $V = X \dot{+} Y$. By the modular law, $W = W \cap V = W \cap (X + Y) = X + (W \cap Y)$. Furthermore, the latter sum is direct since $X \cap Y = 0$. Thus X is a direct summand of W.

(ii) Suppose false. By the minimal condition, we can choose W to be a submodule of V minimal with the property that W is not a finite direct sum of simple submodules. Obviously, $W \neq 0$, so by the minimal condition again, W has a simple submodule X. Part (i) above now implies that $W = X \dot{+} Y$ for some complementary submodule$Y$. But Y is properly smaller than W, so the minimal nature of W implies that Y is a direct sum of finitely many simple submodules. Hence $W = X \dot{+} Y$ also has this property and we have a contradiction. Thus V is a finite direct sum of simple submodules.

3. (i) Note that α is a root of $x^{16} - 3$ and this polynomial is irreducible over the rationals Q by Eisenstein's criterion applied to the prime 3. Thus $|Q[\alpha] : Q| = 16$. Next, α^2 is a root of the irreducible polynomial $x^8 - 3$, by Eisenstein's criterion again, so $|Q[\alpha^2] : Q| = 8$. Since $x^4 - 3$ and $x^2 - 3$ are also irreducible, we conclude that $|Q[\alpha^4] : Q| = 4$ and $|Q[\alpha^8] : Q| = 2$. Thus these intermediate fields are all distinct.

(ii) Now let $Q \subseteq K \subseteq F = Q[\alpha]$. Then $F = K[\alpha]$ and $|F : K| = n$ divides $|F : Q| = 16$. If $f(x) = x^n + \cdots + c$ is the minimal monic polynomial for α over K, then $f(x)$ divides $x^{16} - 3$ and hence the roots of $f(x)$ are all of the form $\alpha\delta$ where δ is a root of unity. Thus since $\pm c$ is the product of the roots of $f(x)$, we see that $c = \alpha^n\epsilon$ where ϵ is also a root of unity. Now $\alpha^n\epsilon = c \in K \subseteq Q[\alpha]$ and α is real, by assumption. Thus ϵ is a real root of unity, so $\epsilon = \pm 1$ and $K \supseteq Q[\alpha^n]$. As we observed, n divides 16 and, by (i) above, $|Q[\alpha^n] : Q| = 16/n$. Since

$$16 = |F : Q| = |F : K| \cdot |K : Q[\alpha^n]| \cdot |Q[\alpha^n] : Q| = n \cdot |K : Q[\alpha^n]| \cdot 16/n,$$

we conclude that $|K : Q[\alpha^n]| = 1$ and hence that $K = Q[\alpha^n]$.

4. We can assume that $X \neq 0$ and we fix a nonzero matrix $x \in X$. By assumption, x is invertible. Now let y be an arbitrary matrix in X and, since the complex numbers are algebraically closed, let $k \in C$ be an eigenvalue of the matrix yx^{-1}. Then

$$\det(kx - y) = \det(kI - yx^{-1})x = \det(kI - yx^{-1}) \cdot \det x = 0$$

by the choice of k. In particular, $kx - y \in X$ is not invertible, so by assumption, $kx - y = 0$. Thus $y = kx \in Cx$ and, since y is arbitrary, $X = Cx$ is one-dimensional.

5. (i) We know that α is the root of a nontrivial polynomial over Q and, by clearing denominators, we can assume that all coefficients are in Z. Say α satisfies

$$z_0 + z_1\alpha + \cdots + z_{s-1}\alpha^{s-1} + n\alpha^s = 0$$

with $z_i \in Z$ and with n a nonzero integer. Multiplying this expression by n^{s-1} yields

$$z_0 n^{s-1} + z_1 n^{s-2}(n\alpha) + \cdots + z_{s-1}(n\alpha)^{s-1} + (n\alpha)^s = 0$$

and hence $n\alpha$ is integral over Z.

(ii) By (i), let n be a nonzero integer with $n\alpha$ integral over Z and choose a prime number p which does not divide n. We claim that $1/p$ is not contained in $Z[\alpha]$. Indeed, if $1/p \in Z[\alpha]$, then by writing this element as an integer polynomial in α, we see that $n^t(1/p)$ is an algebraic integer for some $t \geq 1$. But $n^t(1/p) \in Q$ and Z is integrally closed in Q, so $n^t(1/p) \in Z$. In particular, $p \mid n^t$ and, since p is a prime not dividing n, we have a contradiction. Thus $1/p \notin Z[\alpha]$ and $Z[\alpha] \not\supseteq Q$.