

**Algebra Qualifying Exam**  
**August 2001**

Do all **5** problems.

1. Let  $G$  be a finite group of order  $504 = 2^3 \cdot 3^2 \cdot 7$ .
  - a. Show that  $G$  cannot be isomorphic to a subgroup of the alternating group  $\text{Alt}_7$ . (5 points)
  - b. If  $G$  is simple, determine the number of Sylow 3-subgroups of  $G$ . (5 points)
2. Let  $R$  be a commutative ring with 1 and let  $M$  be a maximal ideal of  $R$ .
  - a. Show that the ring  $R/M^2$  has no idempotents other than 0 and 1. (4 points)
  - b. We know that  $M/M^2$  is naturally an  $R/M$ -module. If  $R$  is Noetherian, prove that this module is finitely generated. (2 points)
  - c. Finally, assume that  $R = K[x_1, x_2, \dots, x_t]$  is a polynomial ring in finitely many variables over the field  $K$ . Prove that  $\dim_K(R/M^2) < \infty$ . (4 points)
3. Let  $F \subseteq E$  be fields and suppose  $0 \neq \alpha \in E$  with  $E = F[\alpha]$ . Assume that some power of  $\alpha$  lies in  $F$  and let  $n$  be the smallest positive integer such that  $\alpha^n \in F$ .
  - a. If  $\alpha^m \in F$  with  $m > 0$ , show that  $m$  is a multiple of  $n$ . (2 points)
  - b. If  $E$  is a separable extension of  $F$ , prove that the characteristic of  $F$  does not divide  $n$ . (4 points)
  - c. If every root of unity in  $E$  lies in  $F$ , show that  $|E : F| = n$ . (4 points)
4. Let  $A$  be a real  $n \times n$  matrix. We say that  $A$  is a *difference of two squares* if there exist real  $n \times n$  matrices  $B$  and  $C$  with  $BC = CB = 0$  and  $A = B^2 - C^2$ .
  - a. If  $A$  is a diagonal matrix, show that it is a difference of two squares. (3 points)
  - b. If  $A$  is a symmetric matrix that is not necessarily diagonal, again show that it is a difference of two squares. (3 points)
  - c. Suppose  $A$  is a difference of two squares, with corresponding matrices  $B$  and  $C$  as above. If  $B$  has a nonzero real eigenvalue, prove that  $A$  has a positive real eigenvalue. (4 points)
5. Let  $K$  be a field of characteristic 0 and view the polynomial ring  $V = K[x]$  as a  $K$ -vector space. Let  $M:V \rightarrow V$  be the linear operator given by multiplication by  $x$ , so that  $M(x^n) = x^{n+1}$  for all integers  $n \geq 0$ . In addition, let  $D:V \rightarrow V$  be the linear operator given by differentiation with respect to  $x$ , so that  $D(x^n) = nx^{n-1}$  for all  $n \geq 0$ . Let  $L$  denote the set of all linear operators of the form  $M^i D^j$  with  $i, j \geq 0$ , where  $M^0 = D^0 = I$  is the identity operator on  $V$ .
  - a. Prove that  $DM - MD = I$ . (3 points)
  - b. Show that  $L$  is a  $K$ -linearly independent set. (4 points)
  - c. For all nonnegative integers  $t$ , prove that  $DM^t$  is in the  $K$ -linear span of the set  $L$ . (3 points)