Algebra Qualifying Exam
August 2002

Do all **5** problems.

1. For any finite group G and prime p, we let $n_p(G)$ denote the number of Sylow p-subgroups of G. Now suppose $K \triangleleft G$, and let P be a Sylow p-subgroup of G.
   a. Show that $n_p(G/K)$ divides $n_p(G)$. (5 points)
   b. Prove that $n_p(G/K) = n_p(G)$ if and only if $P \triangleleft PK$. (5 points)

2. Let R be a commutative ring with 1, and recall that a proper ideal $I \triangleleft R$ is said to be primary if, for all $r, s \in R$, the inclusion $rs \in I$ implies that either $r \in I$ or $s^n \in I$ for some integer $n \geq 1$. Assume now that every proper ideal of R is primary.
   a. If P is a prime ideal of R and if $I \triangleleft R$, prove that either $I \subseteq P$ or $P = IP \subseteq I$. (4 points)
   b. If M is a maximal ideal of R, prove that M is precisely the set of nonunits of R. (3 points)
   c. Show that a proper ideal J of R is prime if and only if, for all $r \in R$, the inclusion $r^2 \in J$ implies that $r \in J$. (3 points)

3. Let F be a field with algebraic closure $\overline{F}$, let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$, and let $E \supseteq F$ be the splitting field of $f(x)$ over F with $E \subseteq \overline{F}$. Assume that $f(x)$ has n distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in E.
   a. Show that there exists an element $\beta \in E$ and n polynomials $p_i(x) \in F[x]$ with $p_i(\beta) = \alpha_i$ for all $i = 1, 2, \ldots, n$. (3 points)
   b. Continuing with the notation of (a), let $g(x) \in F[x]$ be the minimal polynomial of β over F. If $\gamma \in \overline{F}$ is any root of $g(x)$, show that $p_1(\gamma), p_2(\gamma), \ldots, p_n(\gamma)$ are equal to $\alpha_1, \alpha_2, \ldots, \alpha_n$ in some order. (4 points)
   c. Continuing with the notation of (a) and (b), if γ and γ' are both roots of $g(x)$ and if $p_i(\gamma) = p_i(\gamma')$ for all $i = 1, 2, \ldots, n$, show that $\gamma = \gamma'$. (3 points)

4. Let A be an $n \times n$ matrix over an algebraically closed field K.
   a. Show that $A = B + C$ where B is a diagonalizable matrix, C is nilpotent with $C^n = 0$, and $BC = CB$. (4 points)
   b. If $\operatorname{char} K = p > 0$, prove that A^{p^t} is diagonalizable for some integer $t \geq 0$. (2 points)
   c. If K is the complex number field, prove that the exponential matrix $\exp(A) = \sum_{k=0}^{\infty} A^k/k!$ exists. (4 points)

(over)

5. Let R be a ring with 1, let V be a right R-module, and let W be a submodule of V. Suppose that $V = V_1 \dot{+} V_2 \dot{+} \cdots \dot{+} V_n = \sum_i V_i$ is an internal direct sum of the simple (that is, irreducible) submodules $V_1, V_2, \ldots, V_n$. Furthermore, for each subscript j, let $V_j' = \sum_{i \neq j} V_i$ be the internal direct sum of those V_i with $i \neq j$, so that $V = V_j \dot{+} V_j'$.

a. If $W \neq 0$, prove that W contains a minimal proper submodule and a maximal proper submodule. (3 points)

b. If W is a maximal proper submodule of V, prove that there exists a subscript k with $V = W \dot{+} V_k$ and hence that $V/W \cong V_k$. (3 points)

c. If W is simple, show that there exists a subscript j with $W \cong V_j$ and $W \dot{+} V_j' = V$. (4 points)