Algebra Qualifying Exam
August 2003

Do all **5** problems.

1. Let G be a finite group of order $504 = 2^3{\cdot}3^2{\cdot}7$.
   a. Show that G cannot be isomorphic to a subgroup of the alternating group Alt_7. (5 points)
   b. If G is simple, determine the number of Sylow 3-subgroups of G. (5 points)

2. Let R be a commutative integral domain with 1.
   a. Let K be the field of fractions of R, let $t \in R$ be a nonzero element, and suppose that $K = R[1/t]$. In other words, every element of K can be written as a polynomial in $1/t$ with coefficients in R. Show that t is contained in every nonzero prime ideal of R. (5 points)
   b. Now suppose that R is the polynomial ring $R = F[X_1, X_2, \ldots, X_n]$ where F is an infinite field. If $f(X_1, X_2, \ldots, X_n)$ is contained in every nonzero prime ideal of R, show first that $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n \in F$. Then prove that the latter zero-value property implies that f is the zero polynomial. (5 points)

3. Let $F \subseteq E$ be fields and suppose $0 \neq \alpha \in E$ with $E = F[\alpha]$. Assume that some power of α lies in F and let n be the smallest positive integer such that $\alpha^n \in F$.
   a. If $\alpha^m \in F$ with $m > 0$, show that m is a multiple of n. (2 points)
   b. If E is a separable extension of F, prove that the characteristic of F does not divide n. (4 points)
   c. If every root of unity in E lies in F, show that $|E : F| = n$. (4 points)

4. Let A be a real $n \times n$ matrix. We say that A is a *difference of two squares* if there exist real $n \times n$ matrices B and C with $BC = CB = 0$ and $A = B^2 - C^2$.
   a. If A is a diagonal matrix, show that it is a difference of two squares. (3 points)
   b. If A is a symmetric matrix that is not necessarily diagonal, again show that it is a difference of two squares. (3 points)
   c. Suppose A is a difference of two squares, with corresponding matrices B and C as above. If B has a nonzero real eigenvalue, prove that A has a positive real eigenvalue. (4 points)

5. Let K be a field of characteristic 0 and view the polynomial ring $V = K[x]$ as a K-vector space. Let $M\colon V \to V$ be the linear operator given by multiplication by x, so that $M(x^n) = x^{n+1}$ for all integers $n \geq 0$. In addition, let $D\colon V \to V$ be the linear operator given by differentiation with respect to x, so that $D(x^n) = nx^{n-1}$ for all $n \geq 0$. Let L denote the set of all linear operators of the form M^iD^j with $i, j \geq 0$, where $M^0 = D^0 = I$ is the identity operator on V.
   a. Prove that $DM - MD = I$. (3 points)
   b. Show that L is a K-linearly independent set. (4 points)
   c. For all nonnegative integers t, prove that DM^t is in the K-linear span of the set L. (3 points)