Algebra Qualifying Exam
August 2004

Do all **5** problems.

1. Let G be a finite group of order pm, where p is a prime that does not divide m, and let n denote the number of Sylow p-subgroups of G.

a. Show that the exists a homomorphism θ from G to the symmetric group $\mathrm{Sym}(n)$ such that, for all $x \in G$ of order p, the image $\theta(x)$ has exactly one fixed point. (4 points)

b. Now suppose that G is simple and contains an element y of order pq, for some prime $q \neq p$. If θ is as in part (a), show that $\theta(y)$ must contain a cycle of length pq in its cycle decomposition. (3 points)

c. Now let $p = 5$ and suppose that G is a simple group of order 660. Show that G has no element of order 15. (3 points)

2. Let R be a ring with 1, let M be a finitely generated right R-module, and let $N < M$ be a proper submodule of M.

a. Prove that there exists a maximal submodule K of M with $N \subseteq K < M$. (5 points)

b. Show that $N + MJ < M$, where $J = J(R)$ denotes the Jacobson radical of R. (5 points)

3. In the field $\mathbb{C}$ of complex numbers, let $\mathbb{Q}$ be the subfield of rational numbers, let $i = \sqrt{-1}$, and let $\sqrt[4]{2}$ be the positive real fourth root of 2.

a. Prove that the polynomial $X^4 - 2$ is irreducible over the field $\mathbb{Q}[i]$. (4 points)

b. If $\sqrt[4]{2} + i$ is a root of a polynomial $f(X) \in \mathbb{Q}[X]$, show that $i\sqrt[4]{2} + i$ is also a root of that polynomial. (3 points)

c. Compute the degree of the minimal polynomial of $\sqrt[4]{2} + i$ over $\mathbb{Q}$. (3 points)

4. Let V be a vector space of dimension n over a field K. Suppose V is spanned by the $n + 1$ vectors $v_0, v_1, \ldots, v_n$ where $v_0 + v_1 + \cdots + v_n = 0$. Now let W be a second K-vector space and let $w_0, w_1, \ldots, w_n \in W$. Find necessary and sufficient conditions on the elements $w_0, w_1, \ldots, w_n$ so that there exists a linear transformation $T\colon V \to W$ with $T(v_i) = w_i$ for $i = 0, 1, \ldots, n$. (10 points)

5. Let k be a field, let $K = k(x, y)$ be the rational function field over k in the indeterminates x and y, and let $\overline{K}$ denote the algebraic closure of K. Suppose s and t are elements of $\overline{K}$ with $s^2 = x + y$ and $t^3 = xy$, and let $R = k[s, t]$ be the subring of $\overline{K}$ generated by k, s and t. Show that every element $r \in R$ is the root of some irreducible monic polynomial $f(Z) \in K[Z]$ of degree at most 6 with all coefficients in the polynomial ring $k[x, y]$. (10 points)