Algebra Qualifying Exam
August 2006

Do all **5** problems.

1. Let M be a minimal normal subgroup of the finite group G and let N/M be a nontrivial normal subgroup of G/M. Assume that M is a p-group and that N/M is a q-group for some primes p and q, not necessarily distinct.
   a. Show that $G = MH$ where H is a subgroup of G having a nontrivial normal q-subgroup. (4 points)
   b. If M is self-centralizing in G, prove that $p \neq q$. (3 points)
   c. If M is self-centralizing and if H is as in part a, prove that $M \cap H = 1$. (3 points)

2. Let R be a ring with 1, not necessarily commutative. Recall that an element e of R is an idempotent if $e^2 = e$, and an element $0 \neq r \in R$ is a zero divisor if there exists $0 \neq s \in R$ with $rs = 0$ or $sr = 0$. Now assume that R has a nil ideal N such that R/N has no zero divisors.
   a. Show that the only idempotents of R are the elements 0 and 1. (5 points)
   b. If R/N is a division ring, prove that every zero divisor in R is nilpotent. (5 points)

3. Let $\mathbb{C} \supseteq E \supseteq K \supseteq \mathbb{Q}$ be a chain of fields, where $\mathbb{C}$ is the field of complex numbers, $\mathbb{Q}$ is the field of rational numbers, $E = \mathbb{Q}[\alpha]$ with $\alpha^n \in \mathbb{Q}$, and K is generated by all roots of unity in E. Assume that E is a Galois extension of $\mathbb{Q}$.
   a. Show that the Galois group $\mathrm{Gal}(E/K)$ is cyclic. (5 points)
   b. If the restriction τ of complex conjugation to E is in the center of $\mathrm{Gal}(E/\mathbb{Q})$, prove that $|\alpha|^2 \in \mathbb{Q}$, where $|\ |$ denotes complex absolute value. (5 points)

4. Let $V \neq 0$ be a finite dimensional vector space over a field F and let $T\colon V \to V$ be a linear transformation. We say that T is *regular* if its characteristic polynomial and minimal polynomial are equal.
   a. If there exists a vector $v \in V$ such that V is spanned by $v, T(v), T^2(v), \ldots$, prove that T is regular. (5 points)
   b. Assume that T is regular and let W be a subspace of V with $T(W) \subseteq W$. Show that T_W, the restriction of T to W, and $T_{V/W}$, the induced action of T on V/W, are both regular. (5 points)

5. Let $F = \mathrm{GF}(q)$ be the finite field with q elements and let $\mathbf{M}_2(F)$ be the ring of 2×2 matrices over F.
   a. If $A \in \mathbf{M}_2(F)$ has equal eigenvalues in the algebraic closure of F, show that the eigenvalues of A actually belong to F. (4 points)
   b. Determine the number of nonzero nilpotent matrices in $\mathbf{M}_2(F)$ as a function of q. (Hint. Use Jordan canonical form and note that the group G of invertible 2×2 matrices over F has order $(q^2 - 1)(q^2 - q)$.) (6 points)