1. Let $G$ be a finite group of order $|G| = 504 = 2^3 \cdot 3^2 \cdot 7$.
   a. If $G$ has a normal subgroup $N$ of order 8, show that $G$ has at most 8 Sylow 7-subgroups, that is $|\text{Syl}_7(G)| \leq 8$. (5 points)
   b. If $|\text{Syl}_7(G)| \leq 8$, prove that $G$ has an element of order 21. (4 points)
   c. If $G$ is isomorphic to a subgroup of $\text{Sym}_9$, the symmetric group of degree 9, show that $G$ cannot have a normal subgroup of order 8. (1 point)

2. Let $R$ be a commutative integral domain with field of fractions $F$, and assume that $R$ is integrally closed.
   a. Suppose $K$ is a field containing $F$ and let $\alpha \in K$ be integral over $R$. Show that the minimal monic polynomial of $\alpha$ over $F$ is contained in $R[x]$. (5 points)
   b. Let $f(x) \in R[x]$ be a monic polynomial. Show that $f(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. (5 points)

3. Let $F$ be a field of characteristic 0 and let $E$ be a finite Galois extension of $F$.
   a. If $0 \neq \alpha \in E$ with $E = F[\alpha]$, show that $F[\alpha^2] \neq E$ if and only if there exists an automorphism $\sigma \in \text{Gal}(E/F)$ with $\alpha^\sigma = -\alpha$. (6 points)
   b. Prove that there exists an element $\alpha \in E$ with $E = F[\alpha^2]$. (4 points)

4. Let $V$ be a finite-dimensional vector space over the field $F$ with $\dim_F V = n$, and let $(\ , \ ): V \times V \to F$ be a symmetric bilinear form. If $X$ is a subset of $V$, write $X^\perp = \{v \in V \mid (X, v) = 0\}$ for the subspace of $V$ perpendicular to $X$.
   a. If $W$ is a subspace of $V$, show that $\dim_W W + \dim_F W^\perp \geq \dim_F V$. (Hint. If $w \in W$, note that $\{w\}^\perp$ has codimension $\leq 1$ in $V$.) (2 points)
   b. Now suppose $(\ , \ )$ is nonsingular, so that $V^\perp = 0$. If $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ is a basis for $V$, prove that there exists a unique dual basis $\mathcal{A}' = \{a'_1, a'_2, \ldots, a'_n\}$. That is, $\mathcal{A}'$ is a basis with $(a_i, a'_j) = 0$ if $i \neq j$ and $(a_i, a'_i) = 1$. (4 points)
   c. Again suppose $(\ , \ )$ is nonsingular, and let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a second basis for $V$ with dual basis $\mathcal{B}' = \{b'_1, b'_2, \ldots, b'_n\}$. Compare the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$ with the change of basis matrix from $\mathcal{B}'$ to $\mathcal{A}'$. (4 points)

5. Let $R$ be a not necessarily commutative ring with 1.
   a. If $V_1, V_2, \ldots, V_n$ are $n$ nonisomorphic irreducible right $R$-modules, show that there exists an $R$-module epimorphism from $R$, viewed as a right $R$-module, to the external direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_n$. (5 points)
   b. Suppose $R$, viewed as a right $R$-module, has a finite composition series with nonisomorphic composition factors. Prove that the Jacobson radical of $R$ is equal to 0. (5 points)