Algebra Qualifying Exam
August 2008

Do all **5** problems.

1. In this problem we prove that a Sylow 2-subgroup of a simple group of order 168 is its own normalizer.
   a. If G is a group of order 24 and G has a normal Sylow 2-subgroup, show that G contains an element of order 6. (4 points)
   b. If G is a simple group and H is a subgroup of G with $|G:H| = 7$, show that H contains no element of order 6. (3 points)
   c. Let G be a simple group with $|G| = 168$ and let P be a Sylow 2-subgroup of G. Prove that $\mathbb{N}_G(P) = P$. (3 points)

2. Let $\mathbb{Z}$ be the ring of integers and let $S = \mathbb{Z} \oplus \mathbb{Z}$ be the ring external direct sum of two copies of $\mathbb{Z}$. Now let R be the subring of S given by

$$R = \{\, (a,b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \mod 6 \,\}.$$

   a. Show that R is a finitely generated $\mathbb{Z}$-module and conclude that R is a Noetherian ring. (3 points)
   b. Prove that the ideal P of R given by

$$P = \{\, (a,0) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv 0 \mod 6 \,\}$$

   is prime. (2 points)
   c. If Q is a primary ideal of R with $P = \sqrt{Q}$, the radical of Q, show that $Q = P$. (5 points)

3. Let $\mathbb{C}$ denote the complex number field and let $E \subseteq \mathbb{C}$ be the splitting field over the rational numbers $\mathbb{Q}$ of the polynomial $x^3 - 2$.
   a. Show that $|E : \mathbb{Q}| = 6$. (2 points)
   b. If $\alpha \in E$ and $\alpha^5 \in \mathbb{Q}$, prove that $\alpha \in \mathbb{Q}$. (5 points)
   c. Show that there exists $\beta \in E$ with $\beta^2 \in \mathbb{Q}$, but $\beta \notin \mathbb{Q}$. (3 points)

(over)

4. Let S, T and M be $n \times n$ matrices over the complex numbers $\mathbb{C}$ and suppose that $SM = MT$.

a. If $f(x) \in \mathbb{C}[x]$ is the minimal polynomial of T, show that $f(S)M = 0$. (4 points)

b. If $M \neq 0$, deduce that S and T have a common eigenvalue. (3 points)

c. Now suppose $n = 2$,

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}.$$

Find a nonzero matrix M with $SM = MT$ and show that it is impossible to find an invertible matrix M with this property. (3 points)

5. Let R be a subring of the ring $\mathbf{M}_n(\mathbb{C})$ of all complex $n \times n$ matrices, and suppose that R is finitely generated as module over the integers $\mathbb{Z}$. Let $M \in R$.

a. Show that M is contained in a commutative subring S of $\mathbf{M}_n(\mathbb{C})$ that is finitely generated as a $\mathbb{Z}$-module. (3 points)

b. Deduce that there is a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(M) = 0$. (2 points)

c. Prove that $\mathrm{tr}(M)$, the matrix trace of M, is an algebraic integer. (5 points)