1. In this problem we prove that a Sylow 2-subgroup of a simple group of order 168 is its own normalizer.
   a. If $G$ is a group of order 24 and $G$ has a normal Sylow 2-subgroup, show that $G$ contains an element of order 6. (4 points)
   b. If $G$ is a simple group and $H$ is a subgroup of $G$ with $|G : H| = 7$, show that $H$ contains no element of order 6. (3 points)
   c. Let $G$ be a simple group with $|G| = 168$ and let $P$ be a Sylow 2-subgroup of $G$. Prove that $N_G(P) = P$. (3 points)

2. Let $\mathbb{Z}$ be the ring of integers and let $S = \mathbb{Z} \oplus \mathbb{Z}$ be the ring external direct sum of two copies of $\mathbb{Z}$. Now let $R$ be the subring of $S$ given by

   \[ R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv b \mod 6 \}. \]

   a. Show that $R$ is a finitely generated $\mathbb{Z}$-module and conclude that $R$ is a Noetherian ring. (3 points)
   b. Prove that the ideal $P$ of $R$ given by

   \[ P = \{(a, 0) \in \mathbb{Z} \oplus \mathbb{Z} \mid a \equiv 0 \mod 6 \} \]

   is prime. (2 points)
   c. If $Q$ is a primary ideal of $R$ with $P = \sqrt{Q}$, the radical of $Q$, show that $Q = P$. (5 points)

3. Let $\mathbb{C}$ denote the complex number field and let $E \subseteq \mathbb{C}$ be the splitting field over the rational numbers $\mathbb{Q}$ of the polynomial $x^3 - 2$.
   a. Show that $|E : \mathbb{Q}| = 6$. (2 points)
   b. If $\alpha \in E$ and $\alpha^5 \in \mathbb{Q}$, prove that $\alpha \in \mathbb{Q}$. (5 points)
   c. Show that there exists $\beta \in E$ with $\beta^2 \in \mathbb{Q}$, but $\beta \notin \mathbb{Q}$. (3 points)

(over)
4. Let $S$, $T$ and $M$ be $n \times n$ matrices over the complex numbers $\mathbb{C}$ and suppose that $SM = MT$.
   a. If $f(x) \in \mathbb{C}[x]$ is the minimal polynomial of $T$, show that $f(S)M = 0$. (4 points)
   b. If $M \neq 0$, deduce that $S$ and $T$ have a common eigenvalue. (3 points)
   c. Now suppose $n = 2$,
      \[ S = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}. \]
      Find a nonzero matrix $M$ with $SM = MT$ and show that it is impossible to find an invertible matrix $M$ with this property. (3 points)

5. Let $R$ be a subring of the ring $\mathbb{M}_n(\mathbb{C})$ of all complex $n \times n$ matrices, and suppose that $R$ is finitely generated as module over the integers $\mathbb{Z}$. Let $M \in R$.
   a. Show that $M$ is contained in a commutative subring $S$ of $\mathbb{M}_n(\mathbb{C})$ that is finitely generated as a $\mathbb{Z}$-module. (3 points)
   b. Deduce that there is a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(M) = 0$. (2 points)
   c. Prove that $\text{tr}(M)$, the matrix trace of $M$, is an algebraic integer. (5 points)