Algebra Qualifying Exam
August 2009

Do all **5** problems.

1. Let H be a maximal subgroup of the finite group G and let $\mathfrak{X}$ be the set of normal subgroups X of G such that $X \neq 1$ and $X \cap H = 1$.
   a. Show that all members of $\mathfrak{X}$ are minimal normal subgroups of G of the same order. (3 points)
   b. If some member of $\mathfrak{X}$ is abelian, show that all members of $\mathfrak{X}$ are abelian p-groups for some prime p. (3 points)
   c. Let $U, V \in \mathfrak{X}$ be distinct and assume that $\mathfrak{X}$ contains at least one additional member different from U and V. Show that $(UV \cap H) \triangleleft G$ and conclude that $(UV \cap H) \subseteq \mathbb{Z}(UV)$. (4 points)

2. Let $R \subseteq S$ be commutative rings with the same 1, and assume that every element of S is integral over R.
   a. If $r \in R$ has an inverse in S, prove that this inverse is contained in R. (3 points)
   b. Suppose R is a field and let $s \in S$ be a regular element (that is, if $sx = 0$ for some $x \in S$, then $x = 0$). Show that s is invertible in S. (3 points)
   c. If P is a prime ideal of S, prove that P is a maximal ideal of S if and only if $R \cap P$ is a maximal ideal of R. (4 points)

3. Let F be a field and let $f(x) \in F[x]$ be an irreducible polynomial with splitting field E over F. Choose $\alpha \in E$ with $f(\alpha) = 0$. Furthermore, for some fixed integer $n \geq 1$, let $g(x)$ be an irreducible polynomial in $F[x]$ with $g(\alpha^n) = 0$.
   a. Show that $\deg(g)$ divides $\deg(f)$ and that $\deg(f)/\deg(g) \leq n$. (5 points)
   b. If $\deg(f)/\deg(g) = n$ and if the characteristic of F does not divide n, prove that E contains a primitive nth root of unity. (5 points)

4. Let V be a vector space over a field F and let $(\ ,\ ) \colon V \times V \to F$ be a bilinear form. For each $x \in V$ define $A(x) = \{\, y \in V \mid (x, y) = -(y, x) \,\}$. Now suppose v is a fixed element of V with $(v, v) \neq 0$.
   a. For all $x \in V$, show that $A(x)$ is a subspace of V of codimension at most 1. (4 points)
   b. If the characteristic of F is different from 2, prove that $A(v)$ is a subspace of V of codimension precisely 1. (1 point)
   c. If F is algebraically closed and has characteristic different from 2, show that either $(a, a) = 0$ for every element $a \in A(v)$, or there exists $y \in V \setminus A(v)$ with $(y, y) = 0$. (5 points)

5. A multiplicative abelian group A is said to be "divisible" if, for all $a \in A$ and positive integers n, there exists $b \in A$ with $b^n = a$.
   a. If A is divisible and $\overline{A}$ is a homomorphic image of A, prove that $\overline{A}$ is divisible. (2 points)
   b. If A is a finite divisible group, prove that $A = 1$. (3 points)
   c. Suppose A is divisible and that A is a subgroup of the abelian group B. If $A \cap X > 1$ for all nonidentity subgroups X of B, prove that $A = B$. (5 points)