Algebra Qualifying Exam
August 2010

Do all 5 problems.

1. Let G be a finite group and let N be a minimal normal subgroup of G. Suppose $N = S_1 \times S_2 \times \cdots \times S_r$, where each S_i is a simple subgroup and where S_1 is not abelian.

(a) Show that $\mathbf{Z}(N) = 1$, where $\mathbf{Z}(N)$ is the center of N, and deduce that each S_i is nonabelian. (3 points)

(b) If $g \in G$, show that $(S_1)^g = S_i$ for some $i = 1, 2, \ldots, r$. (4 points)

(c) Prove that G has a subgroup of index r. (3 points)

2. Let R be a commutative ring with 1, and let P be a prime ideal of R. Write $\mathcal{M}$ to denote the set of elements of R that are not in P, and recall that $\mathcal{M}$ is a multiplicatively closed subset of R. For each ideal I of R, with I possibly equal to R, define

$$I' = \{\, r \in R \mid rm \in I \text{ for some } m \in \mathcal{M} \,\}.$$

Thus I' is an ideal of R with $I' \supseteq I$.

(a) Prove that $(I')' = I'$. (In other words, the map $I \mapsto I'$ is a closure operator.) Furthermore, show that $I' = R$ if and only if $I \cap \mathcal{M} \neq \emptyset$. (3 points)

(b) If Q is a primary ideal of R, show that either $Q' = Q$ or $Q' = R$. In particular, deduce that $P' = P$. (3 points)

(c) If $P \supseteq I \supseteq P^n$ for some integer $n \geq 1$, prove that I' is a primary ideal with radical equal to P. (4 points)

3. Let $\mathbb{Q}$ be the field of rational numbers and let $f \in \mathbb{Q}[X]$. We say that f is a "special" polynomial provided that f is irreducible in $\mathbb{Q}[X]$, its degree is at least 2, and f splits over $\mathbb{Q}[\alpha]$, where α is some root of f in some extension field of $\mathbb{Q}$.

(a) Let $f \in \mathbb{Q}[X]$ be an irreducible polynomial with degree at least 2, and let L be a splitting field for f over $\mathbb{Q}$. If $\mathrm{Gal}(L/\mathbb{Q})$ is abelian, prove that f is special. (4 points)

(b) Suppose L is a finite degree Galois extension of $\mathbb{Q}$ with L strictly larger than $\mathbb{Q}$. Show that there exists a special polynomial f having a root in L. (3 points)

(c) Prove that the polynomial $X^n - 2 \in \mathbb{Q}[X]$ is not special if $n \geq 3$. (3 points)

(more over)

4. Let V be an n-dimensional vector space over the field $\mathbb{R}$ of real numbers, and let $\theta\colon V \times V \to \mathbb{R}$ be a bilinear form. Given an (ordered) basis $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of V define the $n \times n$ real matrix $M_{\mathcal{A}}$ to be $M_{\mathcal{A}} = [\theta(\alpha_i, \alpha_j)]$.

(a) Show that there exists a nonzero vector $v \in V$ with $\theta(v, V) = 0$ if and only if $M_{\mathcal{A}}$ is a singular matrix. (3 points)

(b) If $\mathcal{B} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ is a second (ordered) basis, we can write $\beta_i = \sum_j p_{i,j}\alpha_j$ for suitable $p_{i,j} \in \mathbb{R}$, where the matrix $P = [p_{i,j}]$ is the change-of-basis matrix. Describe $M_{\mathcal{B}}$ in terms of $M_{\mathcal{A}}$ and the change of basis matrix P. (3 points)

(c) Now assume that θ is symmetric and positive definite. Prove that $M_{\mathcal{A}} = QQ^T$ for some nonsingular matrix Q. Here Q^T is the transpose of Q. (4 points)

5. Let R be a ring (with 1) that is not necessarily commutative, and let M be a right R-module. Suppose that M has a submodule N that is maximal with the property of being noetherian.

(a) Show that no nonzero submodule of M/N is either artinian or noetherian (3 points)

(b) If R is either a right artinian or right noetherian ring, prove that M is noetherian. In other words, show that $M = N$. (3 points)

(c) If $M = R$ viewed as a right R-module, deduce that N is a 2-sided ideal of R. (4 points)