1. Let $G$ be a finite group and let $N$ be a minimal normal subgroup of $G$. Suppose $N = S_1 \times S_2 \times \cdots \times S_r$, where each $S_i$ is a simple subgroup and where $S_1$ is not abelian.
(a) Show that $Z(N) = 1$, where $Z(N)$ is the center of $N$, and deduce that each $S_i$ is nonabelian. (3 points)
(b) If $g \in G$, show that $(S_1)^g = S_i$ for some $i = 1, 2, \ldots, r$. (4 points)
(c) Prove that $G$ has a subgroup of index $r$. (3 points)

2. Let $R$ be a commutative ring with 1, and let $P$ be a prime ideal of $R$. Write $M$ to denote the set of elements of $R$ that are not in $P$, and recall that $M$ is a multiplicatively closed subset of $R$. For each ideal $I$ of $R$, with $I$ possibly equal to $R$, define

$$I' = \{ r \in R \mid rm \in I \text{ for some } m \in M \}.$$ 

Thus $I'$ is an ideal of $R$ with $I' \supseteq I$.
(a) Prove that $(I')' = I'$. (In other words, the map $I \mapsto I'$ is a closure operator.) Furthermore, show that $I' = R$ if and only if $I \cap M \neq \emptyset$. (3 points)
(b) If $Q$ is a primary ideal of $R$, show that either $Q' = Q$ or $Q' = R$. In particular, deduce that $P' = P$. (3 points)
(c) If $P \supseteq I \supseteq P^n$ for some integer $n \geq 1$, prove that $I'$ is a primary ideal with radical equal to $P$. (4 points)

3. Let $\mathbb{Q}$ be the field of rational numbers and let $f \in \mathbb{Q}[X]$. We say that $f$ is a “special” polynomial provided that $f$ is irreducible in $\mathbb{Q}[X]$, its degree is at least 2, and $f$ splits over $\mathbb{Q}[\alpha]$, where $\alpha$ is some root of $f$ in some extension field of $\mathbb{Q}$.
(a) Let $f \in \mathbb{Q}[X]$ be an irreducible polynomial with degree at least 2, and let $L$ be a splitting field for $f$ over $\mathbb{Q}$. If $\text{Gal}(L/\mathbb{Q})$ is abelian, prove that $f$ is special. (4 points)
(b) Suppose $L$ is a finite degree Galois extension of $\mathbb{Q}$ with $L$ strictly larger than $\mathbb{Q}$. Show that there exists a special polynomial $f$ having a root in $L$. (3 points)
(c) Prove that the polynomial $X^n - 2 \in \mathbb{Q}[X]$ is not special if $n \geq 3$. (3 points)

(more over)
4. Let $V$ be an $n$-dimensional vector space over the field $\mathbb{R}$ of real numbers, and let $\theta: V \times V \to \mathbb{R}$ be a bilinear form. Given an (ordered) basis $\mathcal{A} = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$ of $V$ define the $n \times n$ real matrix $M_{\mathcal{A}}$ to be $M_{\mathcal{A}} = [\theta(\alpha_i, \alpha_j)]$.

(a) Show that there exists a nonzero vector $v \in V$ with $\theta(v, V) = 0$ if and only if $M_{\mathcal{A}}$ is a singular matrix. (3 points)

(b) If $\mathcal{B} = \{ \beta_1, \beta_2, \ldots, \beta_n \}$ is a second (ordered) basis, we can write $\beta_i = \sum_j p_{i,j} \alpha_j$ for suitable $p_{i,j} \in \mathbb{R}$, where the matrix $P = [p_{i,j}]$ is the change-of-basis matrix. Describe $M_{\mathcal{B}}$ in terms of $M_{\mathcal{A}}$ and the change of basis matrix $P$. (3 points)

(c) Now assume that $\theta$ is symmetric and positive definite. Prove that $M_{\mathcal{A}} = QQ^T$ for some nonsingular matrix $Q$. Here $Q^T$ is the transpose of $Q$. (4 points)

5. Let $R$ be a ring (with 1) that is not necessarily commutative, and let $M$ be a right $R$-module. Suppose that $M$ has a submodule $N$ that is maximal with the property of being noetherian.

(a) Show that no nonzero submodule of $M/N$ is either artinian or noetherian (3 points)

(b) If $R$ is either a right artinian or right noetherian ring, prove that $M$ is noetherian. In other words, show that $M = N$. (3 points)

(c) If $M = R$ viewed as a right $R$-module, deduce that $N$ is a 2-sided ideal of $R$. (4 points)