Algebra Qualifying Exam
August 2011

Do all 5 problems.

1. Let p be a fixed prime number and let G be a finite group. A normal subgroup K of G is said to be a "normal p-complement" if $p \nmid |K|$ and $|G : K|$ is a power of p.
(a) If G has a normal p-complement and H is a subgroup of G, show that H has a normal p-complement. (3 points)
(b) If G has a normal p-complement and N is a normal subgroup of G, show that G/N has a normal p-complement. (3 points)
(c) Let U and V be normal subgroups of G, and suppose both U and V have normal p-complements. Prove that UV has a normal p-complement. (4 points)

2. Let R be a commutative ring with 1 and let Q be a primary ideal of R. Suppose $a \in R$ with $a \notin Q$ and define $I = \{ r \in R \mid ar \in Q \}$ so that I is an ideal of R.
(a) Show that $\sqrt{I} = \sqrt{Q}$. (3 points)
(b) Prove that I is a primary ideal of R. (3 points)
(c) If R is noetherian, show that the element a can be chosen so that I is a prime ideal. (4 points)

3. Let $K \subseteq F \subseteq E$ be fields with $E = F[\alpha]$ and with $\alpha^n \in F$ for some positive integer n. Suppose K contains a primitive nth root of unity and let L be a field with $K \subseteq L \subseteq E$ and $L \cap F = K$.
(a) If L is Galois over K, show that $L = K[\beta]$ for some element β with $\beta^n \in K$. (7 points)
(b) Show by example that L need not be Galois over K. (Hint. Take $n = 2$.) (3 points)

4. Let n be a positive integer and let V be an n-dimensional vector space over the field K. Let m be a second positive integer and write $V^m = V \times V \times \cdots \times V$ (m times). A "multilinear, alternating" function $f\colon V^m \to K$ is a function $f(x_1, x_2, \ldots, x_m)$ which is multilinear, that is linear in each of the m variables $x_i \in V$, and alternating, that is $f(x_1, x_2, \ldots, x_m) = 0$ if any two of the variables are equal.
(a) Prove that the set S of all multilinear, alternating functions from V^m to K is naturally a K-vector space and that $S = 0$ if $m > n$. (3 points)
(b) If $m \leq n$, prove that $\dim_K S$ is bounded above by the binomial coefficient $\binom{n}{m}$ and that $\dim_K S = 1$ when $m = n$. (7 points)

(more over)

5. Let A be an additive abelian group. We say that A is "free abelian" if it has a subset (possibly infinite) $\mathcal{B} = \{\, b_i \mid i \in \mathcal{I}\,\}$ such that every element $a \in A$ is uniquely writable as a finite sum $a = \sum_{i\in\mathcal{I}} n_i b_i$ where each $n_i \in Z$ is an integer. Next, we say that $a \in A$ is a "divisible element" if for each integer $n > 0$ there exists $a_n \in A$ with $na_n = a$.

(a) If A is free abelian, prove that A contains no nonzero divisible element. (3 points)

(b) Let $A = \prod Z$ be the unrestricted direct product of countably many copies of the additive abelian group Z. Thus A is the set of all countably infinite tuples of elements of the integers Z. Let B be the subgroup of A which is the restricted direct product. Thus B is the set of all countably infinite tuples of elements of Z with almost all entries equal to 0. Prove that A/B contains a nonzero divisible element and conclude that A/B is not free abelian. (7 points)