Algebra Qualifying Exam
August 1991

Do all 5 problems.

1. Let p and q be distinct primes and suppose G is a finite group having precisely $p+1$ Sylow p-subgroups and $q+1$ Sylow q-subgroups. Prove that there exist $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that the subgroup of G generated by P and Q is $PQ = P \times Q$.

2. Let R be a commutative ring with 1. If $a \in R$, we write $\mathrm{ann}(a) = \{ r \in R \mid ar = 0 \}$ for the annihilator of a in R. Thus $\mathrm{ann}(a)$ is an ideal of R and we let $S \subseteq R$ be the set of all elements $a \in R$ such that $\mathrm{ann}(a)$ is a prime ideal of R.

i. (4 points) If R is Noetherian, show that S is nonempty.

ii. (4 points) If $a \in S$ and $r \in R$, show that either $ar = 0$ or $ar \in S$.

iii. (2 points) If $a, b \in S$ and $\mathrm{ann}(a) \neq \mathrm{ann}(b)$, prove that $ab = 0$.

3. Let $F \subseteq E$ be an algebraic extension of fields. We say that an element α of E is *abelian* if $F[\alpha]$ is a Galois extension of F with abelian Galois group $\mathrm{Gal}(F[\alpha]/F)$. Prove that the set of abelian elements of E is a subfield of E containing F.

4. Let V be a finite-dimensional vector space over a field K and let $(\ ,\ )$ be a bilinear form on V. Suppose $T: V \to V$ is a linear transformation satisfying $(v, Tw) = (Tv, w)$ for all $v, w \in V$. Write $N = \ker(T) = \{ v \in V \mid Tv = 0 \}$.

i. (5 points) Assume that the form restricted to N is nondegenerate – that is, if $v \in N$ and $(v, N) = 0$, then $v = 0$. If T is nilpotent, prove that $T = 0$.

ii. (5 points) Find a 2-dimensional example with T a nonzero nilpotent transformation and with the form $(\ ,\ )$ nondegenerate on the whole vector space V.

5. Let R be a ring with 1 and let M be a right R-module. Suppose

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

is a chain of submodules such that, for $i = 1, 2, \ldots, n$, the factors M_i/M_{i-1} are simple and pairwise nonisomorphic. If X and Y are isomorphic submodules of M, prove that $X = Y$.