Algebra Qualifying Exam
August 1992

Do all 5 problems.

1. Let x and y be elements of a finite p-group P and let $z = [x, y]$ be the commutator $x^{-1}y^{-1}xy$ of x and y. Suppose that x lies in every normal subgroup of P which contains z. Prove that $x = 1$.

2. Let $K[x]$ be a polynomial ring over the field K and let R be the subring of $K[x]$ consisting of all polynomials whose x-coefficient is equal to 0. Thus a typical element of R has the form

$$a_0 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

with $a_i \in K$. Show that the principal ideal $(x^2) = x^2R$ is a primary ideal of R which is not equal to a power of its radical.

3. Let E be a finite degree field extension of the rationals $\mathbb{Q}$ and suppose that $f(x)$ is a monic irreducible polynomial in $E[x]$.

i. Show that there exists a unique monic irreducible polynomial $g(x) \in \mathbb{Q}[x]$ such that $f(x)$ divides $g(x)$ in $E[x]$. (4 points)

ii. Let $g(x)$ be as above. If E is a splitting field over $\mathbb{Q}$ for some polynomial in $\mathbb{Q}[x]$, show that the degree of $f(x)$ divides the degree of $g(x)$. (4 points)

iii. Give an example to show that the degree of $f(x)$ need not divide the degree of $g(x)$ in general. (2 points)

4. Let V be a finite dimensional vector space over a field K and let $B\colon V \times V \to K$ be a bilinear form. Suppose that for all $x, y \in V$ we have $B(x, y) = 0$ if and only if $B(y, x) = 0$.

i. If $v, w \in V$ with $B(v, v) \neq 0$, prove that $B(v, w) = B(w, v)$. (5 points)

ii. Deduce that either B is symmetric or $B(v, v) = 0$ for all $v \in V$. (5 points)

5. Let G be the multiplicative group of all 2×2 matrices over the integers $\mathbb{Z}$ whose determinant is equal to 1. Notice that G acts by right multiplication on the set Ω of all 1-dimensional subspaces of the 2-dimensional row space $\mathbb{Q}^2$ over the rational numbers $\mathbb{Q}$.

i. Find all elements of G which act trivially, that is which fix every element of Ω. (4 points)

ii. Prove that G acts transitively. In other words, show that Ω is an orbit under the action of G. (6 points)