Algebra Qualifying Exam
August 1993

Do all 5 problems.

1. Let G be a group of order $|G| = 504 = 7 \cdot 8 \cdot 9$.

i. If there exists an element $x \in G$ of order 21, show that there exists a subgroup $H \subset G$ having index $|G : H| = 8$. Hint. Find the possibilities for n_7, the number of Sylow 7-subgroups of G. (6 points)

ii. If G is simple, show that it contains no element of order 21. (4 points)

2. Let R be a ring and let M be a right R-module which has a composition series. Assume that M has a unique simple submodule N and that N is not isomorphic to any composition factor of M/N. Prove that the ring $\mathrm{End}_R(M)$ is a division ring. In other words, prove that every nonzero R-endomorphism of M is one-to-one and onto.

3. Let $K \subseteq E$ be fields and let $f(x) \in K[x]$ be a polynomial of degree $n \geq 2$ having n distinct roots $a_1, a_2, \ldots, a_n$ in E. Suppose that the field extension $K[a_1, a_2]/K$ has degree $|K[a_1, a_2] : K| = n(n-1)$.

i. Find the degrees of the irreducible factors of $f(x)$ over the field K and over the field $K[a_1]$. (3 points)

ii. If $g(x)$ is the minimal polynomial of $a_1 + a_2$ over K, prove that $a_i + a_j$ is a root of $g(x)$ for all $i \neq j$. Hint. First consider the case $i = 1$. (7 points)

4. Let A be an $m \times n$ matrix over the integers $\mathbb{Z}$ and consider the system of homogeneous linear equations $AX = 0$ where X is the column vector of unknowns $x_1, x_2, \ldots, x_n$. Suppose that every integer solution of this system has all x_i's equal. Prove that the same is true for every real solution of this system of equations.

5. Let $K = \mathbb{Q}[i]$ be the field generated over the rationals $\mathbb{Q}$ by $i = \sqrt{-1}$ and let R be the subring of K defined by

$$R = \{\, a + bi \mid a, b \in \mathbb{Z} \,\}$$

where $\mathbb{Z}$ is the ring of integers. Suppose α is a complex number which is the root of a monic polynomial in $R[x]$. Prove that the minimal monic polynomial of α over K has all coefficients in R.