Algebra Qualifying Exam
August 1994

Do all 5 problems.

1. Let G be a finite groups and let $P \in \mathrm{Syl}_p(G)$ for some prime p. Suppose N is a normal subgroup of G with $|G : N| = |P| > 1$.
   i. Prove that N is the subset of G consisting of all elements of order not divisible by p. (4 points)
   ii. If the elements of G outside of N all have p-power order, prove that P is its own normalizer. (6 points)

2. Let R be a commutative ring and let P be a prime ideal of R. If V is a (right) R-module, define

$$W = \{ v \in V \mid va = 0 \text{ for some } a \in R \setminus P \}$$

where $R \setminus P$ is the set of elements of R not in P.
   i. Show that W is an R-submodule of V. (2 points)
   ii. If R is Noetherian and V is a finitely generated R-module, prove that $Wb = 0$ for some $b \in R \setminus P$. (3 points)
   iii. If V is a simple R-module and $W = 0$, prove that P is a maximal ideal. (5 points)

3. Let E be a splitting field of the polynomial $x^3 - 2$ over the rationals $\mathbb{Q}$, and assume that E is contained in the complex numbers $\mathbb{C}$. Let $F = E \cap \mathbb{R}$ be the real subfield of E, and note that $F = \mathbb{Q}[\sqrt[3]{2}]$.
   i. Show that $G = \mathrm{Gal}(E/\mathbb{Q})$ contains an element σ with the property that the only elements of F fixed by σ are rational. (4 points)
   ii. Let $a \in F$ and suppose that $a^3 \in \mathbb{Q}$. Show that one of a, $a\sqrt[3]{2}$, or $a\sqrt[3]{4}$ is contained in $\mathbb{Q}$. (4 points)
   iii. Prove that $\sqrt[3]{3} \notin E$. (2 points)

4. Let A be an $n \times n$ matrix over a field K and assume that the characteristic polynomial of A has distinct roots in the algebraic closure of K. Prove that any two $n \times n$ K-matrices which commute with A must commute with each other.

5. Let $S = \mathrm{M}_n(F)$ be the ring of $n \times n$ matrices over the field F.
   i. If $s \in S$ is nilpotent, prove that the trace of s is zero. (4 points)
   ii. Suppose R is a ring and that $\theta \colon R \to S$ is a surjective ring homomorphism. Let I be an ideal of R with the property that every element of I is a sum of nilpotent elements of R. Show that $\theta(I) = 0$. (6 points)