Algebra Qualifying Exam
August 1995

Do all **5** problems.

1. Let G be a finite group. We say that a subgroup M of G has property $(*)$ if M is abelian, maximal, and not normal in G.

i. If M and N are distinct subgroups of G with property $(*)$, prove that $M \cap N = Z$, where $Z = \mathbf{Z}(G)$ is the center of G. (2 points)

ii. Let M have property $(*)$ and let $S(M)$ denote the set of all noncentral elements of G which are conjugate to elements of M. Note that

$$S(M) = \bigcup_{x \in G} (M \setminus Z)^x.$$

Compute the cardinality $|S(M)|$ of $S(M)$ in terms of $|M| = m$, $|Z| = z$, and $|G| = g$. Deduce that $g - z > |S(M)| > (g - z)/2$. (5 points)

iii. Show that any two subgroups of G having property $(*)$ must be conjugate in G. (3 points)

2. Let R be a ring. If V and W are right R-modules, we write $V \sim W$ when V is isomorphic to a submodule of W and W is isomorphic to a submodule of V.

i. If $V \sim W$ and if V satisfies the minimum condition, prove that V and W are isomorphic. (4 points)

ii. Suppose $R = \mathbb{Z}$ is the ring of integers. If $V \sim W$ and if V is finitely generated, prove that V and W are isomorphic. (3 points)

iii. Suppose R is a commutative integral domain and let I be a nonzero ideal of R. Show that $R \sim I$ when we view R and I as right R-modules. Conclude that if R is not a PID, then there exist nonisomorphic R-modules V and W with $V \sim W$. (3 points)

3. Let E be the subfield of the real numbers generated over $\mathbb{Q}$ by $\sqrt{2}$ and $\sqrt[3]{2}$.

i. Show that $|E : \mathbb{Q}| = 6$. (2 points)

ii. If K is a field with $\mathbb{Q} \subseteq K \subseteq E$, show that K is one of the fields $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt[3]{2}]$, or E. (5 points)

iii. Prove that $E = \mathbb{Q}[\sqrt{2} + \sqrt[3]{2}]$. (3 points)

4. Let V and W be finite-dimensional vector spaces over an algebraically closed field F and let $A\colon V \to V$ and $B\colon W \to W$ be linear operators. Suppose $T\colon V \to W$ is a *nonzero* linear transformation such that $T(A(v)) = B(T(v))$ for all $v \in V$, and let $N = \ker T$.

i. Show that $A(N) \subseteq N$. (2 points)

ii. Show that there exists $\lambda \in F$ and a vector $v \in V$ with $v \notin N$ such that $A(v) - \lambda v \in N$. (4 points)

iii. If λ is as in part (ii), show that λ is an eigenvalue for both A and B. (4 points)

5. Let S be the set of all 2×2 complex matrices of the form

$$\begin{bmatrix} a & \bar{b} \\ b & \bar{a} \end{bmatrix}$$

with $a, b \in \mathbb{C}$ and where, as usual, $\bar{\ }$ denotes complex conjugation.

i. Show that S is a subring of the ring $\mathrm{M}_2(\mathbb{C})$ of all 2×2 matrices over $\mathbb{C}$. (2 points)

ii. Determine the center Z of S and show that Z is isomorphic to the real numbers $\mathbb{R}$. (3 points)

iii. Prove that

$$I = \left\{ \begin{bmatrix} x & \bar{x} \\ x & \bar{x} \end{bmatrix} \;\middle|\; x \in \mathbb{C} \right\}$$

is a minimal right ideal of S and that it is faithful as a right S-module. (3 points)

iv. Show that $\dim_Z I = 2$ and conclude that $S \cong \mathrm{M}_2(\mathbb{R})$. (2 points)