Algebra Qualifying Exam
August 1996

Do all **5** problems.

1. We say that a group G has property $(*)$ if every normal abelian subgroup of G is contained in $\mathbb{Z}(G)$, the center of G.
   a. Suppose that N and M are normal subgroups of a group G and that G/N and G/M have property $(*)$. Prove that $G/(N \cap M)$ has property $(*)$. (3 points)
   b. Let $N \triangleleft G$ and assume that G/N has property $(*)$. If N has no nontrivial abelian normal subgroups, prove that G has property $(*)$. (3 points)
   c. Show that a finite p-group with property $(*)$ must be abelian. (4 points)

2. Let R be a commutative ring with 1, let $n \geq 2$ be a fixed integer, and suppose that $x^n = x$ for all $x \in R$.
   a. If P is a prime ideal of R, show that R/P is a finite field containing at most n elements. (4 points)
   b. Prove that the intersection of all prime ideals of R is the zero ideal. (2 points)
   c. If R is a Noetherian ring, conclude that R is finite. (4 points)

3. Let $f(x) = x^6 + 3 \in \mathbb{Q}[x]$, let α be a root of $f(x)$ in $\mathbb{C}$, and set $E = \mathbb{Q}[\alpha]$.
   a. Show that E contains a primitive 6^{th} root of unity. (3 points)
   b. Prove that E is Galois over $\mathbb{Q}$. (2 points)
   c. Count the number of intermediate fields F with $\mathbb{Q} \subseteq F \subseteq E$ and $|F : \mathbb{Q}| = 3$. Justify your answer. (5 points)

4. Let V be a finite dimensional vector space over a field K and let $T : V \to V$ be a linear operator. Assume that there exists a nonzero vector $v \in V$ such that V is spanned by the vectors vT^i for $i = 0, 1, 2, \ldots$.
   a. Show that there exists a basis for V with respect to which the matrix of T has the form

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \end{bmatrix}$$

   for suitable $a_i \in K$. (5 points)
   b. Prove that the minimal polynomial and the characteristic polynomial of T are identical. (5 points)

5. The goal of this problem is to prove:

Theorem. *Let* $\mathrm{M}_3(F)$ *denote the space of* 3×3 *matrices over the field* F*. The following are equivalent.*

i. F *has an extension field of degree* 3*.*

ii. $\mathrm{M}_3(F)$ *contains a 3-dimensional subspace whose nonzero members are all invertible matrices.*

iii. $\mathrm{M}_3(F)$ *contains a 2-dimensional subspace whose nonzero members are all invertible matrices.*

a. If E is an extension field of F of degree 3, show that the ring $\mathrm{M}_3(F)$ contains an isomorphic copy of E. Deduce that (i) implies (ii). (5 points)

b. Let A and B be linearly independent invertible matrices in $\mathrm{M}_3(F)$. If the characteristic polynomial of AB^{-1} is not irreducible over F, show that some nonzero F-linear combination of A and B is not invertible. Deduce that (iii) implies (i). (5 points)