Algebra Qualifying Exam
August 1997

Do all **5** problems.

1. Suppose that G is a finite group that has exactly 50 Sylow 7-subgroups. Let $P \in \mathrm{Syl}_7(G)$ and write $N = \mathbb{N}_G(P)$.
  a. Show that N is a maximal subgroup of G. (5 points)
  b. If N has a normal Sylow 5-subgroup Q, prove that $Q \triangleleft G$. (5 points)

2. Let R be a commutative domain with 1.
  a. Let $a, b \in R$ and assume that the principal ideal (ab) is primary. If $a \neq 0$ and b is not a unit of R, prove that $b^n \in (a)$ for some integer $n \geq 1$. (4 points)
  b. Now assume that every principal ideal of R is primary and let P be any nonzero prime ideal of R. Show that P contains every nonunit of R. Deduce that P is the unique nonzero prime ideal of R. (6 points)

3. Let α be a nonzero real number and suppose that $\alpha^n \in \mathbb{Q}$, the rational numbers, for some positive integer n. Let $g(x)$ be the minimal (monic) polynomial of α over $\mathbb{Q}$, and suppose that $\deg g = m$.
  a. Show that $g(0) = \pm\alpha^m$. (5 points)
  b. Deduce that $g(x) = x^m - b$ for some rational number b. (2 points)
  c. Prove that m divides n. (3 points)

4. Let V be a finite dimensional vector space over an algebraically closed field K and let $T : V \to V$ be a linear operator. Also let $I : V \to V$ denote the identity operator. Show that V has a basis consisting of eigenvectors of T if and only if the kernel of $(\lambda I - T)^2$ is equal to the kernel of $\lambda I - T$ for all choices of $\lambda \in K$. (5 points for each direction)

5. Let R be a ring with 1 and let V be a right R-module. Suppose X and Y are R-submodules of V such that $V = X \dot{+} Y$, the (internal) direct sum. If θ is any R-homomorphism from X to Y, define $W_\theta \subseteq V$ to be the set of elements $x - \theta(x) \in V$ for all $x \in X$.
  a. Show that W_θ is an R-submodule of V and that $V = W_\theta \dot{+} Y$. (4 points)
  b. Conversely, suppose U is an R-submodule of V such that $V = U \dot{+} Y$. Prove that $U = W_\theta$ for some R-homomorphism $\theta : X \to Y$. (6 points)