Algebra Qualifying Exam
August 1998

Do all **5** problems.

1. If G is a finite group, we define $\operatorname{soc}(G)$ to be the subgroup generated by all the minimal normal subgroups of G.
   a. If $\langle 1\rangle \neq N \triangleleft G$, show that $N \cap \operatorname{soc}(G) \neq \langle 1\rangle$. (2 points)
   b. Prove that $\operatorname{soc}(\operatorname{soc}(G)) = \operatorname{soc}(G)$. (4 points)
   c. If $\operatorname{soc}(G) = G$, show that every minimal normal subgroup of G is simple. (4 points)

2. Let R be a commutative domain with 1 and let F be its field of fractions. For any element $q \in F$, we define $I_q = \{\, r \in R \mid rq \in R \,\}$.
   a. Show that each I_q is a nonzero ideal of R. (2 points)
   b. If M is a maximal ideal of R, let $R_M = \{\, a/b \in F \mid a \in R,\ b \in R,\ b \notin M \,\}$ and recall that R_M is a subring of F that contains R. Prove that $R = \bigcap_M R_M$, where the latter intersection is over all maximal ideals M of R. (5 points)
   c. Now suppose that $R = \mathbb{Z}[\sqrt{-3}]$ and let $q = (1-\sqrt{-3})/2 = 2/(1+\sqrt{-3}) \in F$. Show that I_q is not a principal ideal. (Hint. Use the fact that the norm map $N(\alpha) = |\alpha|^2$ is multiplicative.) (3 points)

3. Let F be a field and suppose $f(x) \in F[x]$ is an irreducible polynomial. Fix an integer n and let $g(x) = f(x^n)$.
   a. If $h(x)$ is any irreducible factor of $g(x)$ in $F[x]$, show that the degree of $h(x)$ is a multiple of the degree of $f(x)$. (5 points)
   b. Now suppose that F has characteristic 0 and that it contains a primitive nth root of unity. Show that all irreducible factors of $g(x)$ in $F[x]$ have equal degrees. (5 points)

4. Let V be a finite dimensional vector space over the field K and let $(\ ,\ )\colon V \times V \to K$ be a symmetric bilinear form. For any subspace U of V, we let $U^\perp = \{\, v \in V \mid (U, v) = 0 \,\}$. Thus $U^\perp$ is also a subspace of V, and the form is nonsingular precisely when $V^\perp = 0$.
   a. Show that $\dim U + \dim U^\perp \geq \dim V$ for any subspace U of V. (4 points)
   b. If the form is nonsingular and if $V = U + X$ is the sum of subspaces U and X, prove that $\dim U^\perp + \dim X^\perp \leq \dim V$. (2 points)
   c. If the form is nonsingular, show that $\dim U + \dim U^\perp = \dim V$ for any subspace U of V. (4 points)

5. Let A be a (not necessarily finite) abelian group and let B be a subgroup of A.
   a. If B is a direct factor of A, show that B is a direct factor of every subgroup C satisfying $B \subseteq C \subseteq A$. (4 points)
   b. Conversely, assume that B is a direct factor of every subgroup C such that $B \subseteq C \subseteq A$ and C/B is cyclic. If $|A : B| < \infty$, show that B is a direct factor of A. (6 points)