Algebra Qualifying Exam
August 1999

Do all **5** problems.

1. Let G be a group and let $K \subseteq H$ be subgroups of G with $K \triangleleft H$.
   a. Prove that H normalizes $\mathbb{C}_G(K)$. (3 points)
   b. If $H \triangleleft G$ and $\mathbb{C}_H(K) = \langle 1 \rangle$, prove that H centralizes $\mathbb{C}_G(K)$. (7 points)

2. In this problem, the word *ideal* always means *two-sided ideal.* Now let R be a (not necessarily commutative) ring with 1. An ideal P of R is said to be *prime* if, for all ideals A and B of R, the inclusion $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.
   a. If an ideal Q is not prime, show that there exist ideals $A > Q$ and $B > Q$ with $AB \subseteq Q$. (2 points)
   b. Let I and Q be ideals of R and assume that Q is maximal with the property that it contains no power of I. Show that Q is prime. (4 points)
   c. Suppose I is a nonnilpotent ideal of R. If R satisfies the ascending chain condition on ideals, prove that there exists a prime ideal of R which does not contain I. (4 points)

3. Let $K \subseteq L$ be a finite degree extension of fields. Suppose that E and F are intermediate fields, each Galois over K, and that $L = EF$ is the field generated by E and F. (This means that no proper subfield of L contains both E and F.)
   a. Prove that L is Galois over K. (4 points)
   b. If $\mathrm{Gal}(E/K) = G$ and $\mathrm{Gal}(F/K) = H$, show that $\mathrm{Gal}(L/K)$ is isomorphic to a subgroup of $G \times H$. (6 points)

4. In this problem, all matrices are viewed over the complex numbers.
   a. For which complex numbers x, if any, is the matrix $\begin{bmatrix} 1 & -2 \\ 8 & x \end{bmatrix}$ not similar to a diagonal matrix? Explain. (5 points)
   b. Let J be the $n \times n$ matrix all of whose entries are equal to 1. Find a diagonal matrix similar to J or prove that none exists. (5 points)

5. Let $F[x, y]$ be the polynomial ring over the field F in the two indeterminates x and y. Suppose $f(x) \in F[x] \subseteq F[x, y]$ and $g(y) \in F[y] \subseteq F[x, y]$ are polynomials of positive degree in the indeterminates x and y, respectively. Let $I = (f(x), g(y))$ be the ideal of $F[x, y]$ generated by $f(x)$ and $g(y)$.
   a. Prove that $I \neq F[x, y]$. (5 points)
   b. If $f(x) = x - \alpha$ and $g(y) = y - \beta$ for some $\alpha, \beta \in F$, show that I is a maximal ideal of $F[x, y]$. (5 points)