

**Algebra Qualifying Exam**  
**January 2000**

Do all **5** problems.

1. Let  $G$  be a group of order  $2^4 \cdot 5^3 \cdot 11$  and let  $H$  be a group of order  $5^3 \cdot 11$ .
  - a. Show that  $H$  has a normal Sylow 11-subgroup. (2 points)
  - b. If the number of Sylow 5-subgroups of  $G$  is (strictly) less than 16, prove that  $G$  has a proper normal subgroup of order divisible by 5. (4 points)
  - c. If  $G$  has exactly sixteen Sylow 5-subgroups, show that  $G$  has a normal Sylow 11-subgroup. (4 points)
  
2. Let  $R$  be a (not necessarily commutative) ring with 1 and suppose that  $R$  can be written as the sum  $R = \sum_{i=1}^m I_i$ , where the  $I_i$  are finitely many (two-sided) ideals of  $R$  satisfying  $I_i \cap I_j = 0$  whenever  $i \neq j$ .
  - a. Prove that, for every simple right  $R$ -module  $M$ , there exists a unique subscript  $k$  such that  $MI_k \neq 0$ . (5 points)
  - b. Show that if  $i \neq j$ , then every right  $R$ -module homomorphism  $\theta: I_i \rightarrow I_j$  is the zero map. (5 points)
  
3. Let  $L/K$  be a finite degree Galois extension of fields with Galois group given by  $\text{Gal}(L/K) = G$ , and let  $E$  be an intermediate field. Then  $E$  is said to be a 2-tower over  $K$  if there exists a chain of fields  $K = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$  such that  $|E_i : E_{i-1}| = 2$  for all  $i = 1, 2, \dots, n$ .
  - a. If  $G$  is abelian, prove that  $E$  is a 2-tower over  $K$  if and only if the degree  $|E : K|$  is a power of 2. (7 points)
  - b. Show by example that the characterization of 2-towers given in part (a) is false if  $G$  is allowed to be a nonabelian group. (3 points)
  
4. Let  $A$  be an  $n \times n$  matrix over the complex numbers and assume that the rank of  $A$  is equal to 1.
  - a. What are the possible Jordan canonical forms for  $A$ ? Justify your answer. (5 points)
  - b. For each of the forms obtained in part (a), compute the characteristic polynomial of  $A$  and the minimal polynomial of  $A$ . (5 points)
  
5. Let  $R = F[x, y]$  be the polynomial ring over the field  $F$  in the two indeterminates  $x$  and  $y$ , and let  $I = xR$  be the principal ideal of  $R$  generated by  $x$ . Define  $S = F + I$ , so that  $S$  is a subring of  $R$ , and observe that  $I$  is an ideal of  $S$ .
  - a. Show that  $I$  is not finitely generated as an ideal of  $S$ . (5 points)
  - b. Prove that there are infinitely many ideals of  $S$  that are not ideals of  $R$ . (5 points)