Algebra Qualifying Exam
January 2001

Do all **5** problems.

1. Let X and Y be distinct subgroups of a finite group G. We say that X and Y are a weird pair if $|X| = |Y|$ and if no subgroup of G other than X and Y has this same order.

a. If G is a group having a weird pair of subgroups, show that some subgroup of G has a weird pair of normal subgroups. (3 points)

b. If $G = A \times B$ is a direct product of solvable groups, show that the subgroups $A \times 1$ and $1 \times B$ cannot be a weird pair. (4 points)

c. Show that a solvable group cannot contain a weird pair of subgroups. (3 points)

2. Let R be a ring with 1. Recall that an ideal P of R is said to be (right) primitive if there exists a simple right R-module W with $P = \{r \in R \mid Wr = 0\}$. Furthermore, we recall that the Jacobson radical, $\mathrm{Jrad}(R)$, of R is defined to be the intersection of all primitive ideals of R.

a. Let V be a right R-module having a composition series of length n and suppose that R acts faithfully on V. Show that $J = \mathrm{Jrad}(R)$ is an intersection of n primitive ideals of R and that $J^n = 0$. (7 points)

b. Give an example of the situation in part (a) with $n = 2$ and with $\mathrm{Jrad}(R) \neq 0$. Justify your answer. (3 points)

3. Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial with integer coefficients and suppose that $f(\alpha) = 0 = f(2\alpha)$ for some complex number α.

a. Show that $f(0)$, the constant term of f, is not equal to 1. (5 points)

b. If f is irreducible, prove that $\alpha = 0$. (5 points)

4. Let A be an $n \times n$ matrix over the complex numbers and let A^* denote the conjugate transpose of A.

a. Prove that all eigenvalues of the product matrix A^*A are real and nonnegative. (6 points)

b. If I is the $n \times n$ complex identity matrix, show that $\det(I + A^*A)$ is real and positive. (4 points)

5. Let V be a finite-dimensional vector space over some field F and let $T\colon V \to V$ be a linear operator. Write $F[T]$ to denote the ring of all linear operators on V that can be expressed as polynomials in T. Assume that no nonzero proper subspace of V is mapped into itself by T.

a. If $0 \neq S \in F[T]$, show that $\{v \in V \mid vS = 0\}$ is the zero subspace. (3 points)

b. Prove that $F[T]$ is a field. (4 points)

c. Show that $|F[T] : F| = \dim_F V$. (3 points)