Algebra Qualifying Exam
January 2002

Do all **5** problems.

1. Let N be a normal subgroup of the finite group G. A subgroup H of G is said to be a complement for N in G if $NH = G$ and $N \cap H = 1$.
   a. Show that all complements for N in G are isomorphic. (2 points)
   b. If N has a complement in G that is a p-group for some prime p, prove that every Sylow p-subgroup of G contains a complement for N. (3 points)
   c. Assume that the center of N is trivial, that is equal to the identity subgroup, and that every automorphism of N is inner. Prove that N has a unique complement H that is normal in G. (5 points)

2. Let R be a commutative integral domain with 1, and assume that R is integrally closed in its field of fractions. Let $R[x]$ denote the ring of polynomials over R in the variable x.
   a. Let $S \supseteq R$, where S is a commutative integral domain with the same 1, and let $s \in S$ be an element integral over R. If I is the ideal of $R[x]$ consisting of all polynomials $f(x)$ with $f(s) = 0$, prove that I is principal. (5 points)
   b. Let I be a prime ideal of $R[x]$ containing a monic polynomial. If $I \cap R = 0$, prove that I is principal. (5 points)

3. Let K be a field of prime characteristic p and let $F = K(t)$ be the rational function field over K in the indeterminate t. Write $f(x) = x^{2p} - tx^p + t \in F[x]$.
   a. Show that $f(x)$ is an irreducible polynomial in $F[x]$. (3 points)
   b. Let $E = F[s]$, where s is a root of the polynomial $x^p - t \in F[x]$. If L is the splitting field of $f(x)$ over E, prove that $|L : E| \leq 2$. (4 points)
   c. Show that $L = F[\alpha]$, where α is a root of $f(x)$. (3 points)

4. Let V be an n-dimensional vector space over the field K and let $T\colon V \to V$ be a linear operator. Write $K[T]$ to denote the ring of all linear operators on V that can be expressed as polynomials in T, and let $\mathcal{C}$ denote the K-vector space of all linear operators on V that commute with T. Assume that there exists a vector $v_0 \in V$ that is contained in no proper T-invariant subspace of V.
   a. Show that $v_0 K[T] = V$ and deduce that $\dim_K K[T] \geq n$. (3 points)
   a. If $S \in \mathcal{C}$ with $v_0 S = 0$, show that $S = 0$. Deduce that $\dim_K \mathcal{C} \leq n$. (3 points)
   c. Show that $K[T] = \mathcal{C}$, and deduce that the minimal polynomial of T has degree equal to n. (4 points)

(over)

5. Let R be a ring with 1, let V be a right R-module with a composition series, and let $E = \mathrm{End}_R(V)$ be the ring of R-endomorphisms of V.

a. If $\theta\colon V \to V$ is an element of E, prove that θ is one-to-one if and only if it is onto and hence if and only if it is invertible in E. (4 points)

b. If V has a unique minimal R-submodule U, prove that E has a unique maximal ideal I and that every element of $E \setminus I$ is invertible. Be sure to verify that I is indeed an ideal. (3 points)

c. Again, let V have unique minimal submodule U, and suppose in addition that U has multiplicity 1 as an R-composition factor of V. Prove that E is a division ring. (3 points)