Algebra Qualifying Exam
January 2003

Do all **5** problems. In the following, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{Q}$ is the field of rational numbers, and $\mathbb{C}$ is the field of complex numbers.

1. Let N be a normal subgroup of the finite group G and suppose that G/N is a p-group for some prime p.
   a. If $N \subseteq \mathbf{Z}(G)$, the center of G, show that the commutator subgroup G' of G is a p-group. (5 points)
   b. Now assume that N is cyclic (but not necessarily central in G). Prove that $N \cap G' \subseteq \mathbf{Z}(G')$ and deduce that G'' is a p-group. (5 points)

2. Let R be a commutative integral domain with 1. A nonzero, nonunit element $s \in R$ is said to be "special" if, for every element $a \in R$, there exist $q, r \in R$ with $a = qs + r$ and such that r is either 0 or a unit of R.
   a. If $s \in R$ is special, prove that the principal ideal (s) generated by s is maximal in R. (3 points)
   b. Show that every polynomial in $\mathbb{Q}[X]$ of degree 1 is special in $\mathbb{Q}[X]$. (2 points)
   c. Prove that there are no special elements in the polynomial ring $\mathbb{Z}[X]$. (Hint. Apply the definition of special with $a = 2$ and with $a = X$.) (5 points)

3. Let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$, where $F/\mathbb{Q}$ is a finite Galois extension. Let $\alpha \in F$ and let $f(X) \in \mathbb{Q}[X]$ be its minimal monic polynomial. Assume that $1 = |\alpha|$, the absolute value of α, and that $\mathrm{Gal}(F/\mathbb{Q})$ is abelian.
   a. Show that F is closed under complex conjugation. (2 points)
   b. Prove that $|\beta| = 1$ for every complex root β of $f(X)$. (3 points)
   c. Writing $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$, show that $|a_i| \leq 2^n$ for all i with $0 \leq i < n$. (2 points)
   d. Prove that F contains only finitely many algebraic integers having absolute value 1 and deduce that each of these is a root of unity. (3 points)

(over)

4. Let V be vector space over the field K and let $(\ ,\ )\colon V \times V \to K$ be a bilinear form on V.

a. If V is finite dimensional and if W is a proper subspace of V, show that there exists a nonzero vector $v \in V$ with $(w, v) = 0$ for all $w \in W$. (5 points)

b. Now let V have an infinite basis $\mathcal{B}$ and let $(\ ,\ )$ be the unique bilinear form such that, for all $a, b \in \mathcal{B}$, we have $(a, b) = 0$ if $a \neq b$ and $(a, b) = 1$ if $a = b$. If W is the subspace of V spanned by all vectors of the form $a - b$ with $a, b \in \mathcal{B}$, show that W is a proper subspace of V and that there is no nonzero vector $v \in V$ with $(w, v) = 0$ for all $w \in W$. (5 points)

5. Let R be a ring with 1. We say that a right R-module W is "infinitely generated" if it is not finitely generated as an R-module.

a. Let V be a right R-module and let W be a submodule of V. If W is infinitely generated, prove that there exists a submodule M with $W \subseteq M \subseteq V$ such that M is infinitely generated, but such that all submodules of V properly containing M are finitely generated. (5 points)

b. If R is right Noetherian, show that $M = V$ in the above situation. (2 points)

c. If R is not right Noetherian, show that it is possible to choose V and W as in part (a) so that $M \neq V$. (3 points)