Algebra Qualifying Exam
January 2004

Do all **5** problems.

1. Let G be a finite group and let $H \subseteq G$ be a subgroup of index $|G : H| = n$.
   a. Show that $|H : (H \cap H^g)| \leq n$ for all $g \in G$. (2 points)
   b. If H is a maximal subgroup of G and H is abelian, show that $(H \cap H^g) \lhd G$ for all $g \notin H$. (3 points)
   c. Now suppose that G is simple. If H is abelian and n is a prime, prove that $H = 1$. (5 points)

2. Let K be a field and let R be the subring of the polynomial ring $K[X]$ given by all polynomials with X-coefficient equal to 0.
   a. Prove that the elements X^2 and X^3 are irreducible but not prime in the ring R. (5 points)
   b. Show that R is a Noetherian ring, and that the ideal I of R consisting of all polynomials in R with constant term 0 is not principal. (5 points)

3. Recall that a field K is *algebraically closed* if every polynomial $f \in K[X]$ splits over K (is a product of linear factors in $K[X]$). Now let $F \subseteq E$ be an algebraic field extension.
   a. If every polynomial $f(X) \in F[X]$ splits over E, prove that E is algebraically closed. (4 points)
   b. If every polynomial $f(X) \in F[X]$ has a root in E and if F has characteristic 0, prove that E is algebraically closed. (6 points)

4. Let V be a finite dimensional vector space over the field F. Suppose $T\colon V \to V$ is a linear operator and let $f(X) \in F[X]$ be its minimal polynomial.
   a. If $f(X)$ has a nonconstant polynomial factor of degree m, show that V has a nonzero subspace W of dimension $\leq m$ with $T(W) \subseteq W$. (5 points)
   b. Conversely, if V has a nonzero subspace W of dimension n with $T(W) \subseteq W$, show that $f(X)$ has a nonconstant polynomial factor of degree $\leq n$. (5 points)

5. Let R be a ring with 1 and let V be a right R-module. Suppose that $V = X \dot{+} Y$ is the internal direct sum of the two nonzero submodules X and Y.
   a. Show that 0, X, Y and V are the only R-submodules of V if and only if X and Y are nonisomorphic simple R-modules. (6 points)
   b. If X and Y are nonisomorphic simple R-modules, prove that $\mathrm{End}_R(V)$, the ring of R-endomorphisms of V, is isomorphic to the direct sum of two division rings. (4 points)