Algebra Qualifying Exam - January 2005

Do all 5 problems. Show all work.

1. Let G be a finite group with $|G| = 660 = 2^2{\cdot}3{\cdot}5{\cdot}11$ and suppose that $E \subseteq G$ is a subgroup of order 11. Assume that $\mathbf{C}_G(E) = E$.
(a) Prove that $|\mathbf{N}_G(E)| = 55$. (3 points)
(b) If $M \triangleleft G$, show that either $E \subseteq M$ or $|M| \equiv 1 \bmod 11$. (3 points)
(c) Show that every minimal normal subgroup of G contains E. (4 points)

2. All rings in this problem are commutative with 1. A ring S is said to be finitely generated if there exist finitely many elements $s_1, s_2, \ldots, s_n \in S$ such that every element of S can be written as a sum of products of these generators. Now let R be a ring, let G be a finite group of automorphisms of R, and let $R^G = \{r \in R \mid r^g = r \text{ for all } g \in G\}$ be the fixed subring.
(a) If $r \in R$, prove that R^G contains a finitely generated subring T such that r is integral over T. (4 points)
(b) If R is finitely generated, show that R^G contains a finitely generated subring S such that R is integral over S. (2 points)
(c) Let R and S be as in (b). Deduce that R is a finitely generated S-module and hence that R^G is a finitely generated S-module. Conclude that R^G is a finitely generated ring. (Hint. You can use the fact that any finitely generated ring is a homomorphic image of a polynomial ring in finitely many variables over the integers and hence is a Noetherian ring.) (4 points)

3. Let F be a field and let $f(X) \in F[X]$ be an irreducible polynomial. Suppose $E \supseteq F$ is an extension field of F containing a root α of $f(X)$ satisfying $f(\alpha^2) = 0$. Show that $f(X)$ splits over E. (10 points)

4. Let F be an algebraically closed field and let $\mathrm{M}_n(F)$ be the ring of $n \times n$ matrices over F. Describe those matrices $X \in \mathrm{M}_n(F)$ with the property that all matrices that commute with X are diagonalizable. (10 points)

5. An additive abelian group U is said to be uniform if, for every two nonzero subgroups X and Y, we have $X \cap Y \neq 0$. Let us also say that U is max-uniform if U is uniform and if U is not contained in any properly larger uniform group.
(a) If U is uniform and has a nonzero element of finite order, show that there exists a prime p such that every element of U has order a power of p. (3 points)
(b) Let A be an abelian group and let U be a uniform subgroup. Suppose M is a subgroup of A maximal with the property that $M \cap U = 0$. Show that A/M is a uniform group. (3 points)
(c) Let A be an abelian group and let U be a max-uniform subgroup. Prove that there exists a subgroup M of A with $A = U \dot{+} M$, the internal direct sum of U and M. Include details of the Zorn's Lemma argument. (4 points)