

Algebra Qualifying Exam - January 2006

Do all 5 problems. Show all work.

- Let A , B and K be minimal normal subgroups of the group G with $K \neq A$, $K \neq B$ and $K \subseteq AB$.
 - Show that $KA = AB = KB$. (4 points)
 - Prove that $A \cong K \cong B$. (3 points)
 - Show that AB is abelian. (3 points)
- Let $\mathbb{Z}[x]$ be the polynomial ring over the integers \mathbb{Z} in the indeterminate x . Let R be the subring of $\mathbb{Z}[x]$ consisting of all polynomials having their coefficients of x and x^2 equal to 0.
 - Prove that $\mathbb{Q}(x)$ is the field of fractions of R , where \mathbb{Q} is the field of rational numbers. (2 points)
 - Find the integral closure of R in $\mathbb{Q}(x)$. (4 points)
 - Does there exist a polynomial $g(x) \in R$ such that R is generated as a ring by 1 and $g(x)$? (4 points)
- Let n be a positive integer and let F be a field of characteristic not dividing n . Let $f(x) \in F[x]$ be the polynomial $x^n - a$ for some $0 \neq a \in F$ and let E be a splitting field for $f(x)$ over F .
 - Show that E contains a primitive n th root of unity ε . (3 points)
 - If $\varepsilon \in F$, show that all irreducible factors of $f(x)$ in $F[x]$ have the same degree and that $|E : F|$ divides n . (3 points)
 - Now assume that n is a power of 2, but do not assume that $\varepsilon \in F$. Prove that $|E : F|$ is a power of 2. (4 points)
- Let V be a finite-dimensional vector space over the real numbers \mathbb{R} .
 - If $\dim_{\mathbb{R}} V$ is odd, prove that every linear operator $A: V \rightarrow V$ has at least one real eigenvalue. (3 points)
 - Suppose A_1, A_2, \dots, A_n are finitely many pairwise commuting linear operators on V . Assume that none of the operators A_i has a negative real eigenvalue. If the sum $A_1 + A_2 + \dots + A_n$ is equal to the negative of the identity operator on V , show that $\dim_{\mathbb{R}} V$ is even. (Hint. Use induction on the dimension of V .) (7 points)
- Let R be a ring with 1 and let M be a right R -module. We say that the module M has property $(*)$ if every nonzero homomorphic image of M has a simple submodule.
 - If M is generated by its artinian submodules, show that M has property $(*)$. (5 points)
 - If M has property $(*)$ and is noetherian, show that it is artinian. (5 points)