Algebra Qualifying Exam - January 2006

Do all 5 problems. Show all work.

1. Let A, B and K be minimal normal subgroups of the group G with $K \neq A$, $K \neq B$ and $K \subseteq AB$.
(a) Show that $KA = AB = KB$. (4 points)
(b) Prove that $A \cong K \cong B$. (3 points)
(c) Show that AB is abelian. (3 points)

2. Let $\mathbb{Z}[x]$ be the polynomial ring over the integers $\mathbb{Z}$ in the indeterminant x. Let R be the subring of $\mathbb{Z}[x]$ consisting of all polynomials having their coefficients of x and x^2 equal to 0.
(a) Prove that $\mathbb{Q}(x)$ is the field of fractions of R, where $\mathbb{Q}$ is the field of rational numbers. (2 points)
(b) Find the integral closure of R in $\mathbb{Q}(x)$. (4 points)
(c) Does there exist a polynomial $g(x) \in R$ such that R is generated as a ring by 1 and $g(x)$? (4 points)

3. Let n be a positive integer and let F be a field of characteristic not dividing n. Let $f(x) \in F[x]$ be the polynomial $x^n - a$ for some $0 \neq a \in F$ and let E be a splitting field for $f(x)$ over F.
(a) Show that E contains a primitive nth root of unity ε. (3 points)
(b) If $\varepsilon \in F$, show that all irreducible factors of $f(x)$ in $F[x]$ have the same degree and that $|E : F|$ divides n. (3 points)
(c) Now assume that n is a power of 2, but do not assume that $\varepsilon \in F$. Prove that $|E : F|$ is a power of 2. (4 points)

4. Let V be a finite-dimensional vector space over the real numbers $\mathbb{R}$.
(a) If $\dim_{\mathbb{R}} V$ is odd, prove that every linear operator $A\colon V \to V$ has at least one real eigenvalue. (3 points)
(b) Suppose $A_1, A_2, \ldots, A_n$ are finitely many pairwise commuting linear operators on V. Assume that none of the operators A_i has a negative real eigenvalue. If the sum $A_1 + A_2 + \cdots + A_n$ is equal to the negative of the identity operator on V, show that $\dim_{\mathbb{R}} V$ is even. (Hint. Use induction on the dimension of V.) (7 points)

5. Let R be a ring with 1 and let M be a right R-module. We say that the module M has property $(*)$ if every nonzero homomorphic image of M has a simple submodule.
(a) If M is generated by its artinian submodules, show that M has property $(*)$. (5 points)
(b) If M has property $(*)$ and is noetherian, show that it is artinian. (5 points)