Algebra Qualifying Exam
January 2007

Do all **5** problems.

1. Let G be a finite group and let $\mathrm{Syl}_p(G)$ denote its set of Sylow p-subgroups.
   a. Suppose that S and T are distinct members of $\mathrm{Syl}_p(G)$ chosen so that $S \cap T$ is maximal among all such intersections. Prove that the normalizer $\mathbf{N}_G(S \cap T)$ has more than one Sylow p-subgroup. (5 points)
   b. Show that $S \cap T = 1$ for all $S, T \in \mathrm{Syl}_p(G)$, with $T \neq S$, if and only if $\mathbf{N}_G(P)$ has exactly one Sylow p-subgroup for every nonidentity p-subgroup P of G. (5 points)

2. Let R be a commutative, Noetherian integral domain.
   a. If P is a prime ideal of R, show that the radical of P^n is P. (2 points)
   b. If R has a unique nonzero prime ideal P, prove that all ideals of R are primary. (3 points)
   c. Conversely, let us now assume that all ideals of R are primary, and let P and Q be distinct prime ideals of R with $Q \not\supseteq P$. Since $P^n \cap Q$ is primary, deduce first that $P^n \supseteq Q$ and then that $Q = 0$. (Hint. Consider whether the intersection $P^n \cap Q$ can be irredundant.) (5 points)

3. Let F be a field of characteristic 0 and let $f \in F[X]$ be an irreducible polynomial of degree > 1 with splitting field $E \supseteq F$. Define $\Omega = \{\alpha \in E \mid f(\alpha) = 0\}$.
   a. Let $\alpha \in \Omega$ and let m be a positive integer. If $g \in F[X]$ is the minimal polynomial of α^m over F, show that $\{\beta^m \mid \beta \in \Omega\}$ is the set of roots of g. (3 points)
   b. Now fix $\alpha \in \Omega$ and suppose that $\alpha r \in \Omega$ for some $r \in F$. Show that, for all $\beta \in \Omega$ and integers $i \geq 0$, we have $\beta r^i \in \Omega$. Conclude that r is a root of unity. (3 points)
   c. If α and r are as in (b) and if m is the multiplicative order of the root of unity r, show that $f(X) = g(X^m)$, where g is the minimal polynomial of α^m over F. (4 points)

4. Let V be a finite dimensional vector space over a field K and assume that V is endowed with a not necessarily symmetric bilinear form $\langle\ ,\ \rangle\colon V \times V \to K$. We Let R and L denote the right and left radicals of $\langle\ ,\ \rangle$ given by $R = \{x \in V \mid \langle V, x\rangle = 0\}$ and $L = \{x \in V \mid \langle x, V\rangle = 0\}$, so that these are both subspaces of V.
   a. Use the bilinear form to construct a linear transformation T from V to the dual space $(V/R)^*$ of V/R such that $\ker(T) = L$. (6 points)
   b. Show that $\dim_K L = \dim_K R$, and deduce that the map T is surjective. (4 points)

5. Let A be an additive abelian group and let B be a subgroup. We say that B is essential in A, and write $B \operatorname{ess} A$, if and only if $B \cap X \neq 0$ for all nonzero subgroups X of A.
   a. If $B_1 \operatorname{ess} A_1$ and $B_2 \operatorname{ess} A_2$, prove that $(B_1 \oplus B_2) \operatorname{ess} (A_1 \oplus A_2)$. (5 points)
   b. If $B \operatorname{ess} A$, and B has no nonzero elements of finite order, prove that A has no nonzero elements of finite order. (2 points)
   c. Let Q denote the additive group of rational numbers and suppose that $Q \operatorname{ess} A$, for some abelian group A. Prove that $Q = A$. (3 points)