Algebra Qualifying Exam
January 2008

Do all **5** problems.

1. Let G be a finite nonabelian group with center Z.
   a. If G/Z is a p-group, for some prime p, show that G has a normal Sylow p-subgroup and that p divides $|Z|$. (5 points)
   b. If G/Z is solvable, show that G has a nonidentity normal p-subgroup for some prime dividing $|G : Z|$. (5 points)

2. Let $R \subseteq S$ be commutative rings with the same 1, and suppose that S is finitely generated as an R-module.
   a. If an element $r \in R$ is not invertible in R, prove that it is not invertible in S. HINT. If r is invertible in S, consider a polynomial in $R[X]$ having $1/r$ as a root. (5 points)
   b. If the ideals of R satisfy the ascending chain condition, show that the ideals of S satisfy the ascending chain condition. (5 points)

3. Working in the field of complex numbers, let ε be a primitive 16^{th} root of unity, and let $\alpha = \varepsilon\sqrt{2}$. Set $E = \mathbb{Q}[\varepsilon]$, where $\mathbb{Q}$ is the field of rational numbers, let $f(X) = X^8 + 16 \in \mathbb{Q}[X]$, and note that α is a root of $f(X)$.
   a. Show that $\sqrt{2} \in \mathbb{Q}[\varepsilon^2]$. (3 points)
   b. Conclude that $f(X)$ splits in $E[X]$. (2 points)
   c. If $G = \text{Gal}(E/\mathbb{Q})$, prove that no nonidentity element of G fixes α. Conclude that $f(X)$ is irreducible in $\mathbb{Q}[X]$. (5 points)

4. Let V be a finite-dimensional vector space over the field F of characteristic $p > 0$, let $T\colon V \to V$ be a linear operator on V, and set $W = \{v \in V \mid vT = v\}$. Suppose that $T^p = I$, the identity, and that $\dim_F W = 1$.
   a. Show that $(T - I)^p = 0$ and conclude that $\dim_F V \leq p$. (4 points)
   b. If $\dim_F V < p$, prove that $(T - I)^{p-1} = 0$. (3 points)
   c. If, for some vector $v \in V$, we have $v + vT + vT^2 + \cdots + vT^{p-1} \neq 0$, prove that $\dim_F V = p$. (3 points)

5. Let R be a ring with 1. A (right) R-module V is said to be strongly n-generated, for some integer n, if every submodule of V is generated as an R-module by some set of $\leq n$ elements.
   a. If V is strongly n-generated and if W is a submodule of V, prove that both W and V/W are strongly n-generated. (3 points)
   b. Let W be a submodule of V. If W is strongly n-generated and if V/W is strongly m-generated, prove that V is strongly $(n + m)$-generated. (5 points)
   c. If V has composition length n, prove that V is strongly n-generated. (2 points)