Algebra Qualifying Exam
January 2009

Do all 5 problems.

1. Let G be a finite group of order $p(p+1)$, where p is an odd prime, and assume that G does not have a normal Sylow p-subgroup.

(a) Find (with proof) the number of elements of G with order different from p. (3 points)

(b) Show that each nonidentity conjugacy class of elements with order different from p has size at least p, and conclude that there is precisely one such conjugacy class. (5 points)

(c) Prove that $p+1$ is a power of 2. (2 points)

2. Let $\mathbb{R}$ be the field of real numbers and let $\mathbb{C} \supseteq \mathbb{R}$ be the complex field. Define S to be the subring of the polynomial ring $\mathbb{C}[X]$ consisting of all polynomials with real constant term so that

$$S = \mathbb{R} + \mathbb{C}X + \mathbb{C}X^2 + \mathbb{C}X^3 + \cdots .$$

(a) Show that the ideal of S consisting of all polynomials with 0 constant term is not principal. (4 points)

(b) Let I be a nonzero ideal of S and choose $0 \neq f \in I$ to have minimal possible degree n. If $g \in I$, show that there exists $s \in S$ with $g - sf$ either equal to 0 or to a polynomial of degree n. Conclude that I is generated by f and perhaps one additional polynomial of degree n. (6 points)

3. Let F be the field $\mathrm{GF}(p)$ of prime order $p > 2$ and suppose that the polynomial $f(X) = X^m + 1 \in F[X]$ is irreducible.

(a) Show that every root of f in a splitting field of the polynomial has multiplicative order $2m$. (4 points)

(b) Prove that $2m$ divides $p^m - 1$, but that $2m$ does not divide $p^n - 1$ for any integer n with $0 < n < m$. (3 points)

(c) Show that $m \neq 4$. (3 points)

4. Let V be a finite-dimensional vector space over the complex numbers $\mathbb{C}$ and let $T\colon V \to V$ be a linear operator on V.

(a) If T is diagonalizable on V and if W is a subspace of V with $T(W) \subseteq W$, prove that T is diagonalizable on W. (6 points)

(b) If T has the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

with respect to some basis of V, decide (with proof) whether T is diagonalizable on V. (4 points)

5. In the following, all groups are additive abelian groups, and recall that an abelian group is said to be noetherian if its set of subgroups satisfies the ascending chain condition or equivalently the maximal condition. Furthermore, a nonzero group is said to be uniform if it contains no direct sum of nonzero subgroups.

(a) Show that every nonzero noetherian group contains a nonzero uniform subgroup. (4 points)

(b) Suppose $G = U \dotplus V$ is the internal direct sum of the two subgroups U and V with U uniform. If G contains the direct sum $A \dotplus B$ with A and B both nonzero, prove that $(A \dotplus B) \cap V \neq 0$. (3 points)

(c) Let $G = U \dotplus V$ be as above with U uniform. If G contains the direct sum $A \dotplus B \dotplus C$ with A, B and C all nonzero, prove that V is not uniform. (3 points)