1. Let $S_7$ denote the symmetric group on seven points, and let $A_7$ be the corresponding alternating group.
   (a) Find the number of elements of order 7 in $S_7$, and find the order of the centralizer in $S_7$ of one of these elements. (3 points)
   (b) Find the order of the normalizer of a Sylow 7-subgroup in $A_7$. (3 points)
   (c) Prove that $S_7$ does not contain a simple subgroup $G$ of order $504 = 2^33^27$. (4 points)

2. Let $E \supseteq K$ be fields with $|E : K| < \infty$ and let $R$ be a subring (with 1) of $K$ having $K$ as its field of fractions.
   (a) Prove that there exists a ring $S$ with $R \subseteq S \subseteq E$ such that $S$ is a finitely generated $R$-module and such that $E$ is the field of fractions of $S$. (5 points)
   (b) Let $\alpha \in E$ be integral over $R$. If $R$ is integrally closed in $K$, prove that the minimal monic polynomial $f(X) \in K[X]$ of $\alpha$ over $K$ has all its coefficients in $R$. (5 points)

3. Let $F \subseteq E$ be finite fields, where $|F| = q < \infty$ and $|E : F| = n$.
   (a) Prove that every monic irreducible polynomial in $F[X]$ of degree dividing $n$ is the minimal polynomial over $F$ of some element of $E$. (4 points)
   (b) Compute the product of all the monic irreducible polynomials in $F[X]$ of degree dividing $n$. (2 points)
   (c) Suppose $|F| = 2$. Determine the number of monic irreducible polynomials of degree 10 in $F[X]$. (4 points)

4. Let $V$ be a finite dimensional vector space over the field $F$ and let $T : V \to V$ be a linear operator on $V$ with characteristic polynomial $f(X) \in F[X]$.
   (a) Show that $f(X)$ is irreducible in $F[X]$ if and only if there are no proper nonzero subspaces $W$ of $V$ with $T(W) \subseteq W$. (6 points)
   (b) If $f(X)$ is irreducible in $F[X]$ and if the characteristic of $F$ is 0, show that $T$ is diagonalizable when we extend the field $F$ to its algebraic closure. (4 points)

5. Let $R$ be a ring (with 1) and let $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = R$ be a chain of right ideals of $R$ such that each of the $n$ quotients $V_i = I_i/I_{i-1}$ is a simple right $R$-module.
   (a) If $M$ is a maximal right ideal of $R$, prove that $R/M$ is isomorphic as a right $R$-module to some $V_i$. (3 points)
   (b) Now assume that the $V_i$’s are pairwise nonisomorphic $R$-modules. Prove that the intersection of all the maximal right ideals of $R$ is equal to 0. (5 points)
   (c) Continue to assume that the $V_i$’s are pairwise nonisomorphic $R$-modules and deduce that $R$ is a finite ring direct sum of division rings. (2 points)