Algebra Qualifying Exam
January 2010

Do all 5 problems.

1. Let S_7 denote the symmetric group on seven points, and let A_7 be the corresponding alternating group.
 (a) Find the number of elements of order 7 in S_7, and find the order of the centralizer in S_7 of one of these elements. (3 points)
 (b) Find the order of the normalizer of a Sylow 7-subgroup in A_7. (3 points)
 (c) Prove that S_7 does not contain a simple subgroup G of order $504 = 2^3 3^2 7$. (4 points)

2. Let $E \supseteq K$ be fields with $|E : K| < \infty$ and let R be a subring (with 1) of K having K as its field of fractions.
 (a) Prove that there exists a ring S with $R \subseteq S \subseteq E$ such that S is a finitely generated R-module and such that E is the field of fractions of S. (5 points)
 (b) Let $\alpha \in E$ be integral over R. If R is integrally closed in K, prove that the minimal monic polynomial $f(X) \in K[X]$ of α over K has all its coefficients in R. (5 points)

3. Let $F \subseteq E$ be finite fields, where $|F| = q < \infty$ and $|E : F| = n$.
 (a) Prove that every monic irreducible polynomial in $F[X]$ of degree dividing n is the minimal polynomial over F of some element of E. (4 points)
 (b) Compute the product of all the monic irreducible polynomials in $F[X]$ of degree dividing n. (2 points)
 (c) Suppose $|F| = 2$. Determine the number of monic irreducible polynomials of degree 10 in $F[X]$. (4 points)

4. Let V be a finite dimensional vector space over the field F and let $T\colon V \to V$ be a linear operator on V with characteristic polynomial $f(X) \in F[X]$.
 (a) Show that $f(X)$ is irreducible in $F[X]$ if and only if there are no proper nonzero subspaces W of V with $T(W) \subseteq W$. (6 points)
 (b) If $f(X)$ is irreducible in $F[X]$ and if the characteristic of F is 0, show that T is diagonalizable when we extend the field F to its algebraic closure. (4 points)

5. Let R be a ring (with 1) and let $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = R$ be a chain of right ideals of R such that each of the n quotients $V_i = I_i/I_{i-1}$ is a simple right R-module.
 (a) If M is a maximal right ideal of R, prove that R/M is isomorphic as a right R-module to some V_i. (3 points)
 (b) Now assume that the V_i's are pairwise nonisomorphic R-modules. Prove that the intersection of all the maximal right ideals of R is equal to 0. (5 points)
 (c) Continue to assume that the V_i's are pairwise nonisomorphic R-modules and deduce that R is a finite ring direct sum of division rings. (2 points)