Algebra Qualifying Exam
January 2011

Do all 5 problems.

1. Let the group $G = H \times K$ be the (internal) direct product of its subgroups H and K. Suppose that there exists a group X and surjective homomorphisms $\theta\colon H \to X$ and $\phi\colon K \to X$. Then we let

$$U = \{\, hk \in G \mid h \in H,\ k \in K,\ \text{and } \theta(h) = \phi(k) \,\}.$$

(a) Show that U is a subgroup of G such that $UH = G = UK$, $U \cap H = \ker\theta$ and $U \cap K = \ker\phi$. (5 points)
(b) If V is a subgroup of G with $V \supseteq U$, show that both $V \cap H$ and $V \cap K$ are normal subgroups of G. (2 points)
(c) If X is a simple group, prove that U is a maximal subgroup of G that contains neither H nor K. (3 points)

2. Let R be a commutative ring with 1, let $(a) = aR$ be the principal ideal of R generated by $a \in R$ and let P be a prime ideal of R properly contained in (a).
(a) Show that $P = aP$. (4 points)
(b) Now assume that P is a finitely generated ideal and prove that there exists $b \in R$ with $(1 - ab)P = 0$. (Hint. Here $1 - ab$ is the determinant of an appropriate matrix with coefficients in R.) (5 points)
(c) In particular, if R is a domain conclude that either $P = 0$ or $(a) = R$. (1 point)

3. Let $F \subseteq E$ be fields of characteristic 0, and suppose that $E = F[\alpha]$, where $\alpha^p \in F$ for some prime p. Now let $E^* = E[\varepsilon]$ where ε is a primitive p-th root of 1.
(a) Show that E^* is a Galois extension of F. (3 points)
(b) If E is Galois over F, prove that $E = F$ or $E = E^*$. (5 points)
(c) Show, by example, that it is possible to have $E = E^*$ even if F does not contain ε. (2 points)

(over)

4. Let V be a finite-dimensional vector space over the complex numbers $\mathbb{C}$ and let $T\colon V \to V$ be a linear operator on V.

(a) Suppose W is a subspace of V with $T(W) \subseteq W$. Then the restriction S of T to W is a linear operator on W. Prove that the characteristic polynomial $f_S(x)$ of S (on W) divides the characteristic polynomial $f_T(x)$ of T (on V). (4 points)

(b) Let λ be a root of $f_T(x)$ of multiplicity m and let $V_\lambda = \{ v \in V \mid T(v) = \lambda v \}$, so that V_λ is a subspace of V. Prove that $1 \leq \dim_{\mathbb{C}} V_\lambda \leq m$. (4 points)

(c) Find an example of a vector space V, a linear operator T and a root λ of $f_T(x)$ such that λ has multiplicity 5 as a root of the polynomial, but $\dim_{\mathbb{C}} V_\lambda = 1$. (2 points)

5. Let R be a (noncommutative) ring with 1 and suppose that R has a unique maximal right ideal M.

(a) Show that M is a two-sided ideal of R. (3 points)

(b) Prove that every element of $R \setminus M$ has a two-sided inverse. (5 points)

(c) Show that 0 and 1 are the only idempotent elements of R. (2 points)